

# Timed pushdown automata and branching vector addition systems

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Highlights 2017  
London

## Results summary

Two models/problems

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trPDA  
(non-emptiness)

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1-BVASS<sup>±</sup>  
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decidable

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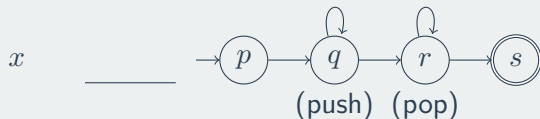
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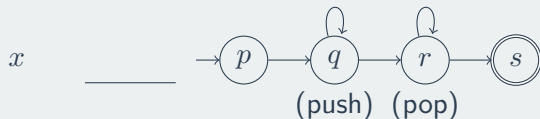


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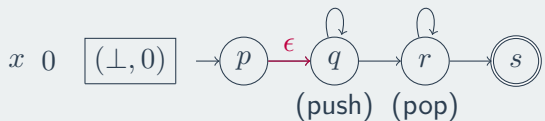
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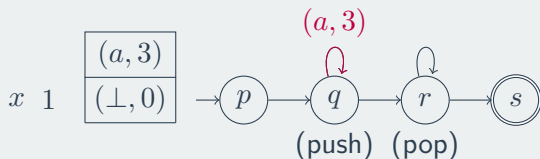


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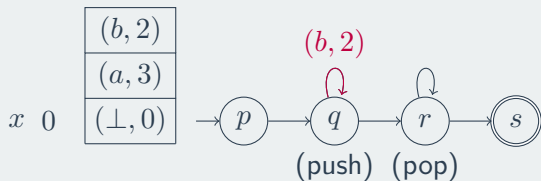
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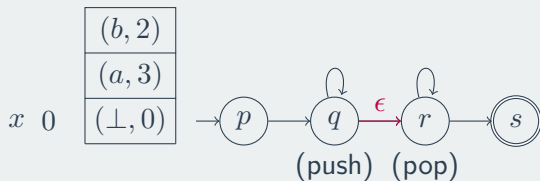
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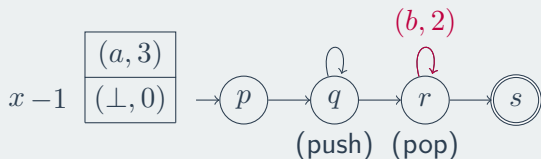


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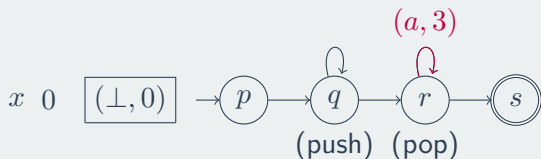
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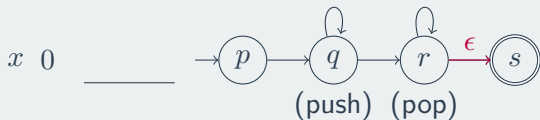
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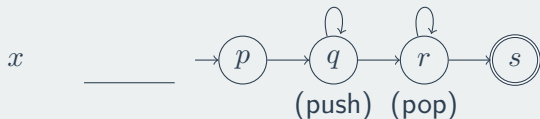
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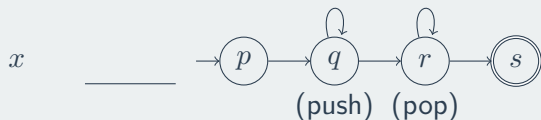
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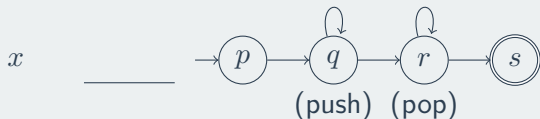
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- An attempt to combine recursion and time:

[Bouajjani, Echahed, Robbana], [Abdulla, Atig, Stenman]

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Timed stack – this talk

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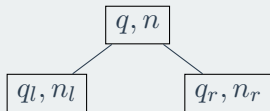
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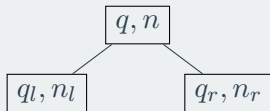
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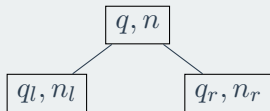
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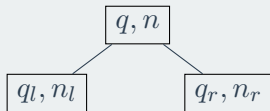
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1-BVASS (no subtraction) state of art etc:  
previous talk (I hope)

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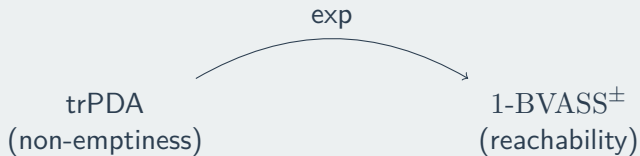
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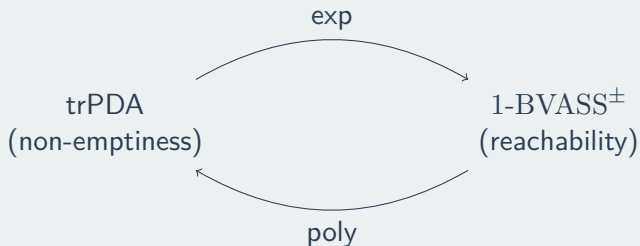
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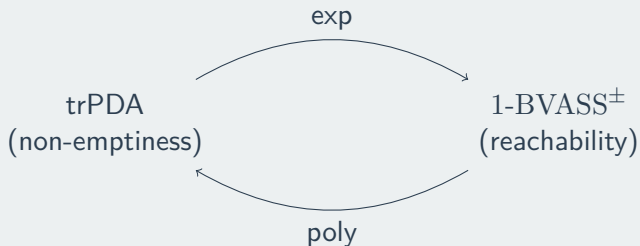
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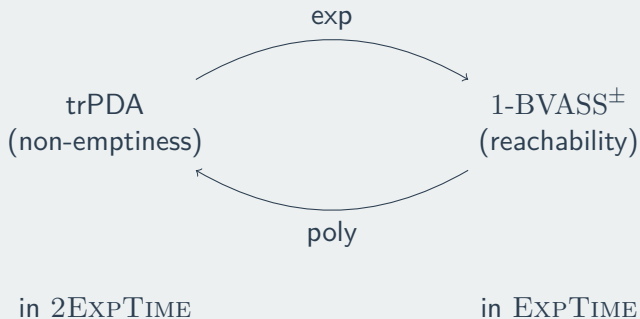
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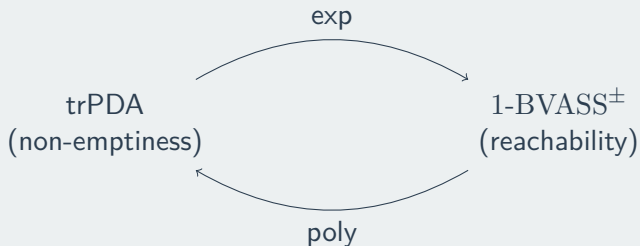
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in  $2\text{EXPTIME}$

$\text{EXPTIME}$ -hard

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$\text{PSPACE}$ -hard

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