On the complexity of quantified integer programming

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Highlights 2017
September 13, 2017

Quantified Integer Programming (QIP)

Example (with 3 quantifier blocks):

$$\exists\, x\ \forall\, y\ \exists\, z:\ \ A \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} \leq \boldsymbol{c}$$

$x, y, z \in \mathbb{N}$

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Our results: Complexity of QIP

Theorem

QIP with $\boldsymbol{k}$ quantifier blocks is complete for $\boldsymbol{k}^{\text{th}}$ level of $\mathbf{PH}$.

Same complexity as quantified SAT:

$$\exists\, x \;\forall\, y \;\exists\, z : \;(x \vee \neg y) \,\&\, (y \vee \neg z) \,\&\, (z \vee \neg x \vee y),$$

$x, y, z \in \{0, 1\}$.

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SAT $\leq$ IP:

- Standard
- Works in the quantified setting (albeit some tricks needed)

Quantified IP $\leq$ Quantified SAT:

- **Small model property**

Background

$k = 1$: IP is $\mathbf{NP}$-complete

- Upper bound: via minimal solutions to linear integer inequalities [von zur Gathen, Sieveking (1978)]
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[Haase, CSL-LICS'14]

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- **Small model property** (our main contribution)

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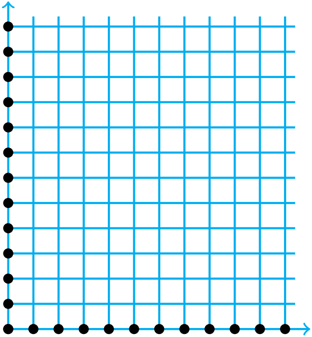
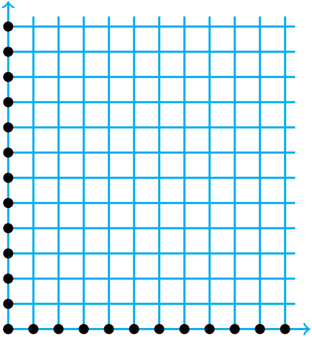
$$
\begin{aligned}
&\{(x, y) : \exists\, z_1\ \forall\, z_2\ \exists\, z_3.\ \varphi(x, y, z_1, z_2, z_3)\} \\
= &\{(x, y) : (x = 0) \text{ or } (y = 0)\}
\end{aligned}
$$

?





Thank you!