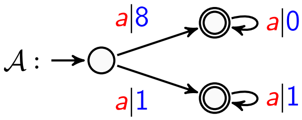
# Bounded-Delay Determinization of Max-Plus Automata

Ismaël Jecker

September 13, 2017

# Definitions

Max-Plus Automaton



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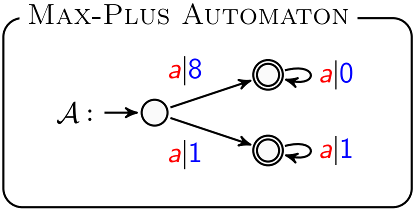


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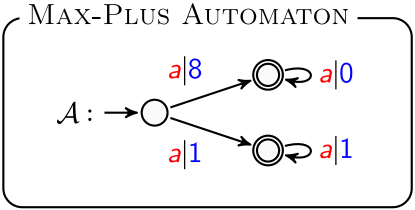


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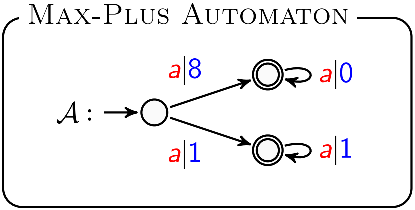


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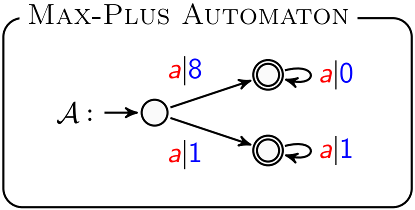


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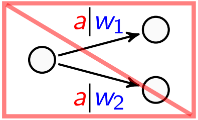

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Input: A max-plus automaton $\mathcal{A}$.

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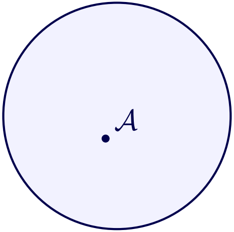


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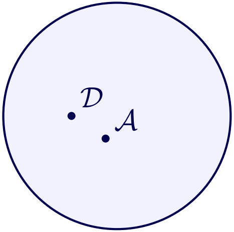


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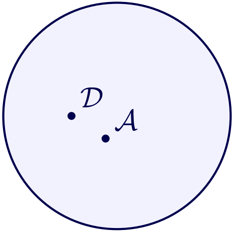


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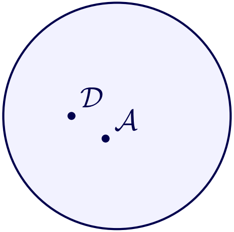


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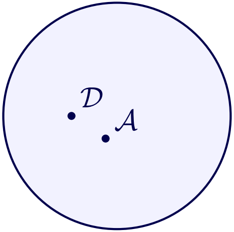


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$\exists$ polynomial $p_{\mathcal{A}} : \mathbb{N} \to \mathbb{N}$ s.t. $\forall w \in \Sigma^*$, $\mathcal{A}$ admits at most $p_{\mathcal{A}}(|w|)$ runs on $w$.

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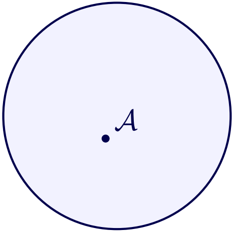


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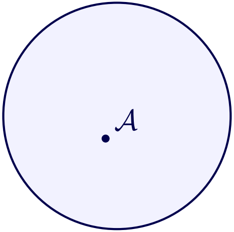


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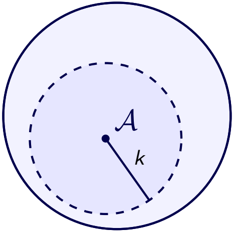


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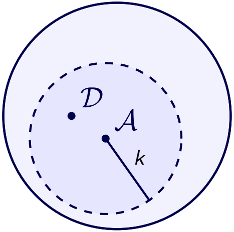


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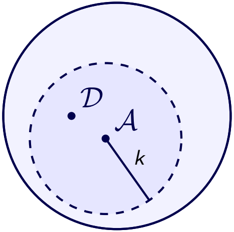


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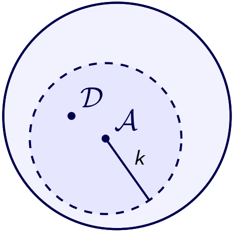


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Reduction to 0-regret.

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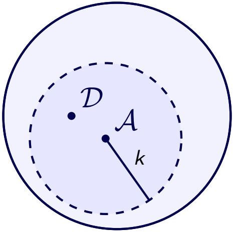


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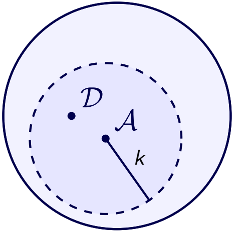


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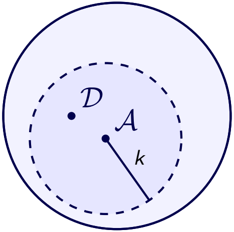


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If $\mathcal{A}$ is determinizable, then it is $k$-delay determinizable for some $k$.

$\mathcal{D}$ deterministic $\Rightarrow \text{delay}_{\mathcal{A}}(\mathcal{D}) \neq \infty$

# Delay Distance

$\textsc{Delay}_{\mathcal{A}}(\mathcal{B})$

Minimal integer $k$ such that for every accepting run $\rho$ of $\mathcal{B}$, there is an accepting run of $\mathcal{A}$ $k$-close to $\rho$ with same input.

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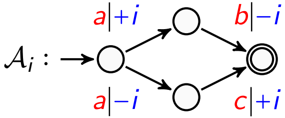

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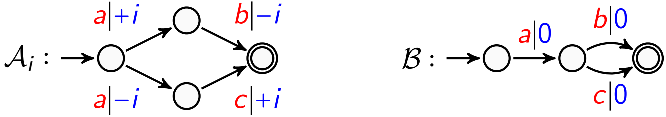

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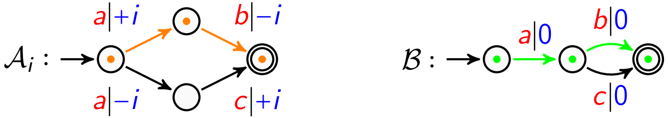


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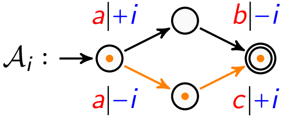


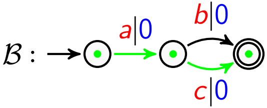


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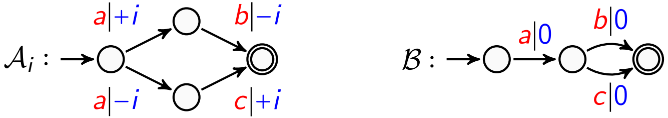


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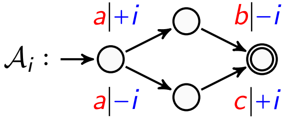


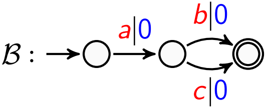


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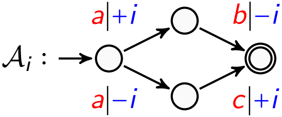


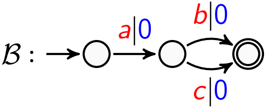


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- $\mathcal{A}_i$ is $i$-delay determinizable;
- For every $k < i$, $\mathcal{A}_i$ is not $k$-delay determinizable.

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[D. Kirsten, S. Lombardy, 2009]

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For more details

Emmanuel Filiot, J. , Nathan Lhote, Guillermo Pérez, Jean-François Raskin.
On Delay and Regret Determinization of Max-Plus Automata. LICS 2017