The Theory of Languages

Highlights in London

September 12-15, 2017

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Universal laws

$$a \cup b = b \cup a \qquad \text{(commutativity of union)}$$

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c \qquad \text{(associativity of concatenation)}$$

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Variables in X

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Mirror image

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Intersection

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Language equivalence

$$\mathrm{Lang} \models e \simeq f \text{ iff } \forall\Sigma, \forall\sigma : X \to \mathcal{P}(\Sigma^\star), \widehat{\sigma}(e) = \widehat{\sigma}(f).$$

representation

$$r : \mathbb{E}_X \to R$$

Effective representation

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Computable (→ r)

Decidable (→ R)

Effective free representation

$$r : \mathbb{E}_X \to R$$

Computable

Decidable

$$r(e) = r(f) \iff \mathrm{Lang} \models e \simeq f$$

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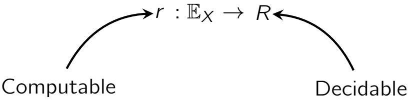
Computable (→ r) Decidable (→ R)

$$\textcolor{red}{Ax} \vdash e = f \iff r(e) = r(f) \iff \mathrm{Lang} \models e \simeq f$$

Effective free representation



$$\color{red}{Ax} \vdash e = f \Longleftrightarrow r(e) = r(f) \Longleftrightarrow \mathrm{Lang} \models e \simeq f$$

Kleene Algebra, KA with Tests, Kleene lattices, Allegories, Monoids,...

Example

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Compare terms over $\langle \cdot, \cap \rangle$ assuming that certain variables contain ε.

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Tests

Adapting tools from relation algebra

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- $s, t ::= a \mid s \cdot t \mid s \cap t$

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- **Adding mirror image:** duplicate the alphabet.

Main results

$$e, f \in \mathbb{E}_X ::= 0 \mid 1 \mid a \mid e \cup f \mid e \cap f \mid e \cdot f \mid e^{\smile} \mid e^{\star}.$$

Decidability theorem

The equational theory of languages over $\mathbb{E}_X$ is ExpSpace-complete.

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$$e, f \in \mathbb{E}_X^{fin} ::= 0 \mid 1 \mid a \mid e \cup f \mid e \cap f \mid e \cdot f \mid e^{\smile}.$$

Axiomatisability theorem

The equational theory of languages over $\mathbb{E}_X^{fin}$ is finitely axiomatisable.

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$$b \leq a \setminus c \qquad \Leftrightarrow \qquad a \cdot b \leq c \qquad \Leftrightarrow \qquad a \leq c \,/\, b$$

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That's all folks!

See more at:
`http://paul.brunet-zamansky.fr`

Outline

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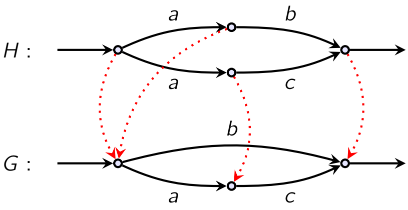

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Corollary

$$\mathrm{Lang} \models u \subseteq v \Leftrightarrow \mathcal{G}(u) \blacktriangleleft \mathcal{G}(v).$$

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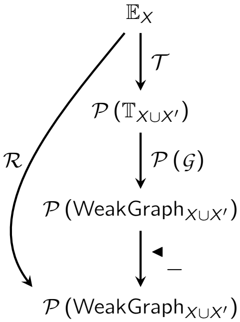

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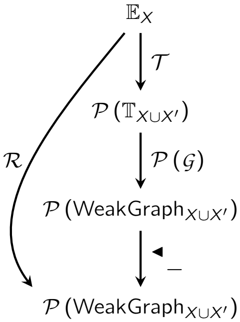

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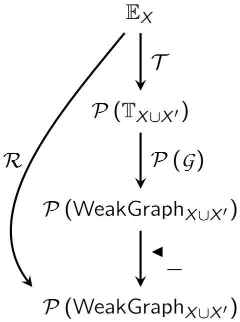

Lemma

If e doesn't use the Kleene star, then $\mathcal{T}(e)$ is finite.

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Decidability and Complexity

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Main Theorem

The equational theory of languages over the signature $\mathbb{E}_X$ is ExpSpace-complete.

Axiomatization for series parallel terms

$u, v, w \in \mathbb{SP}_X ::= a \mid u \parallel v \mid u \cdot v, \quad A \subseteq X.$

$$\frac{}{A \vdash_{sp} u \leq u} \qquad \frac{A \vdash_{sp} u \leq v \qquad A \vdash_{sp} v \leq w}{A \vdash_{sp} u \leq w}$$

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Associativity

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Meet-semilattice

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