# Nondeterminism does not make regular separability harder

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Highlights 2017, Warsaw

languages of **finite words**

Fix a class of languages $C$

**Regularity problem for** $C$

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**Input**: a languages from $C$

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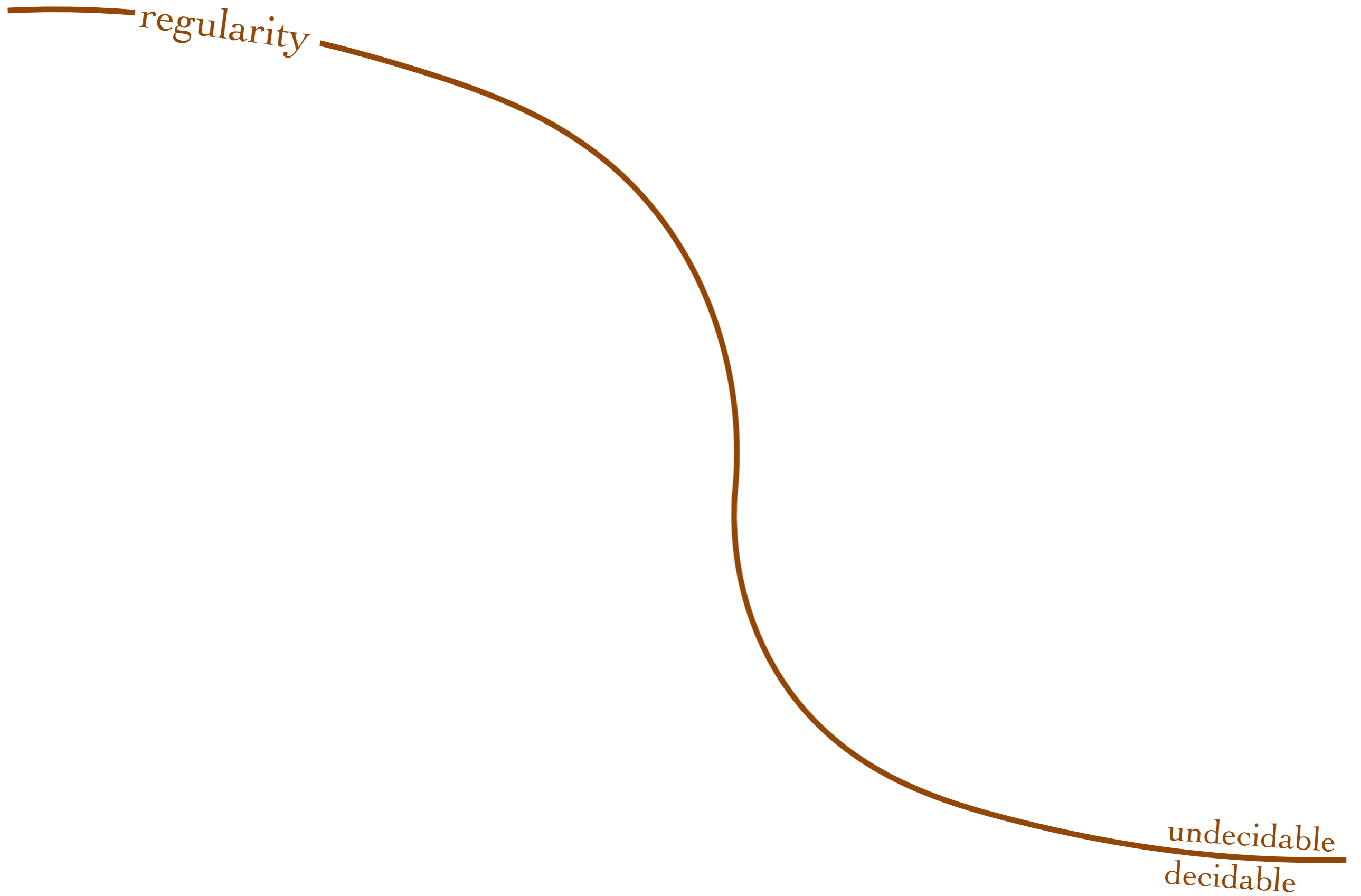
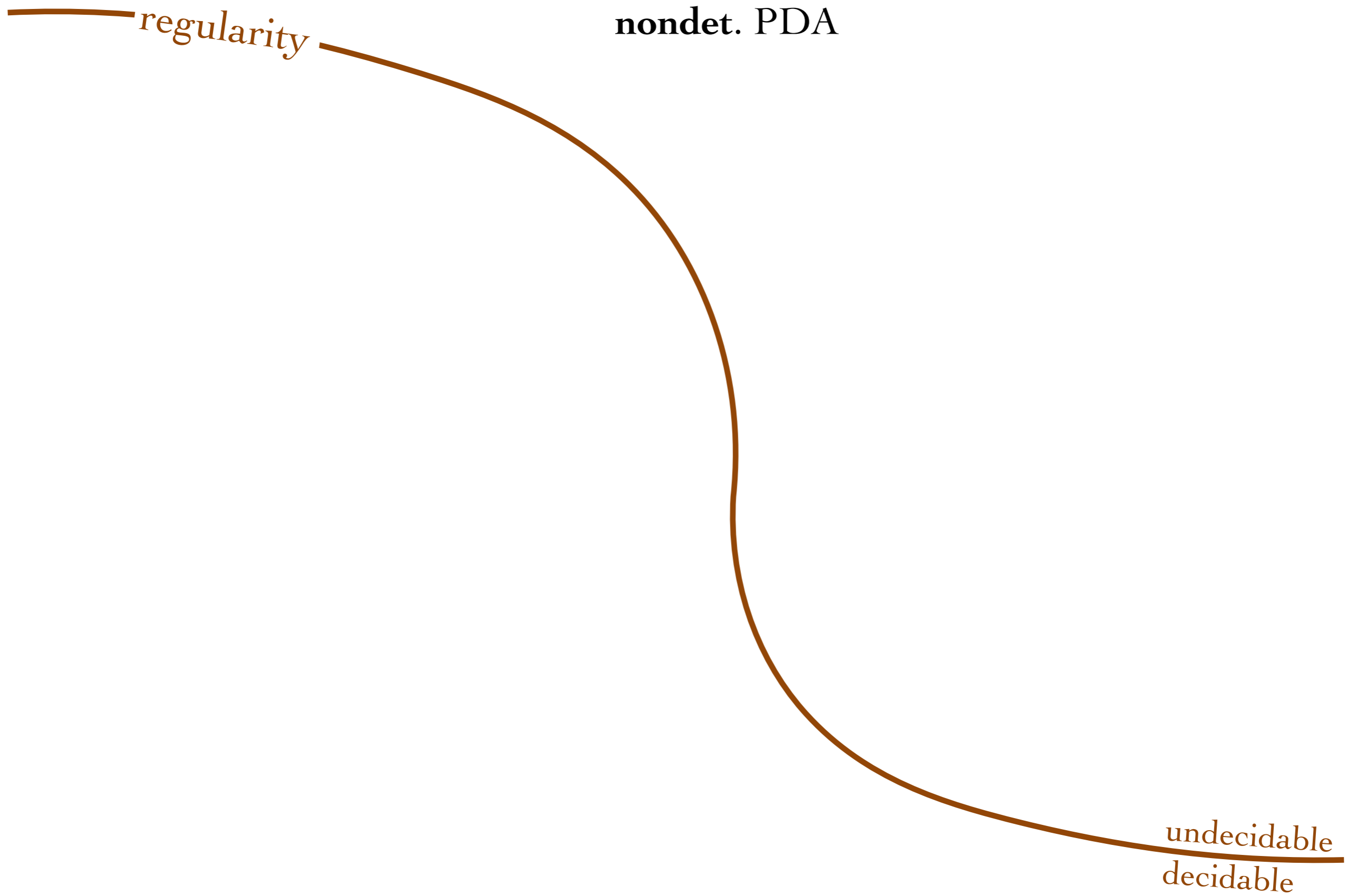**Question**: is this language regular?

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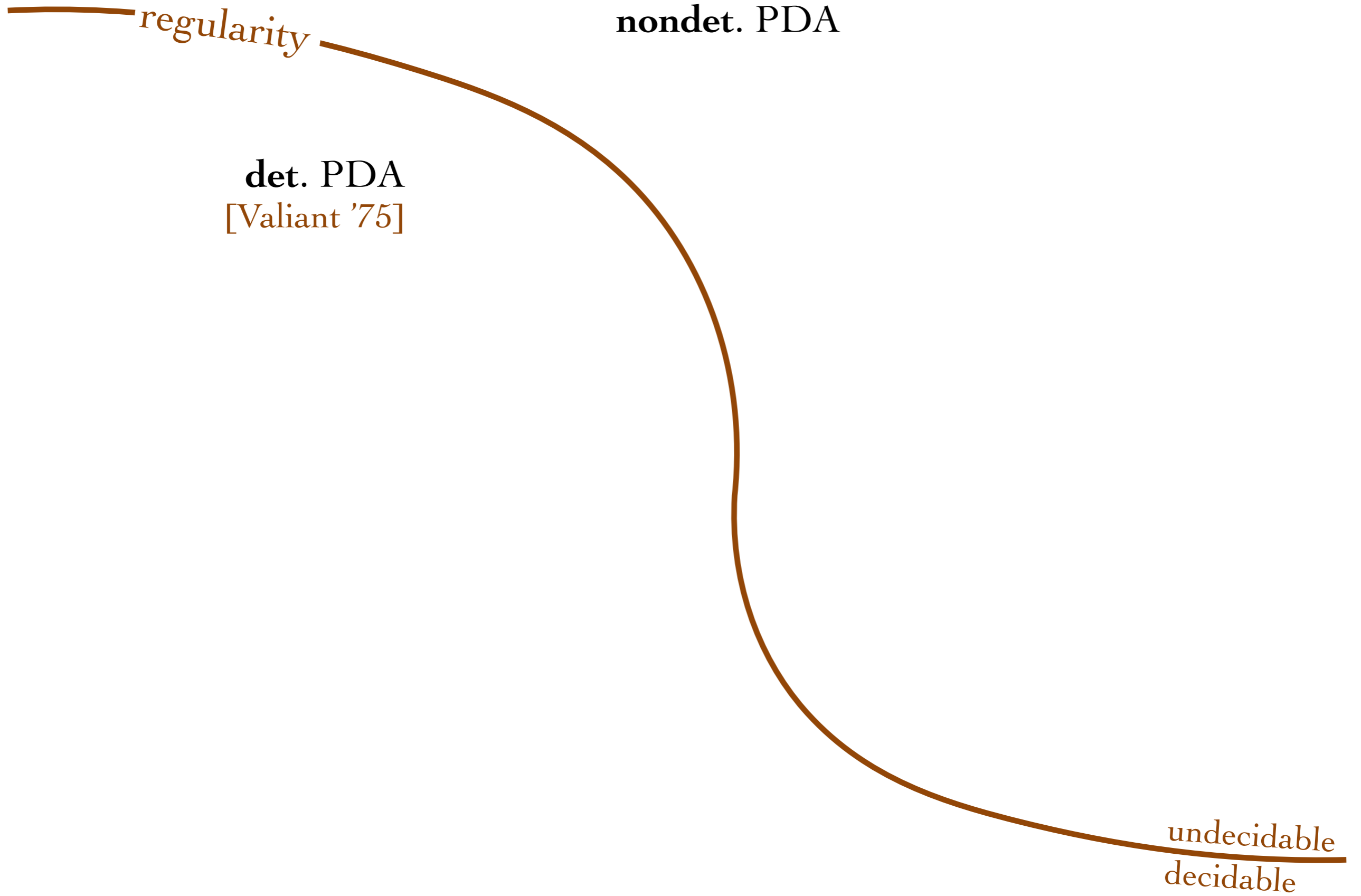
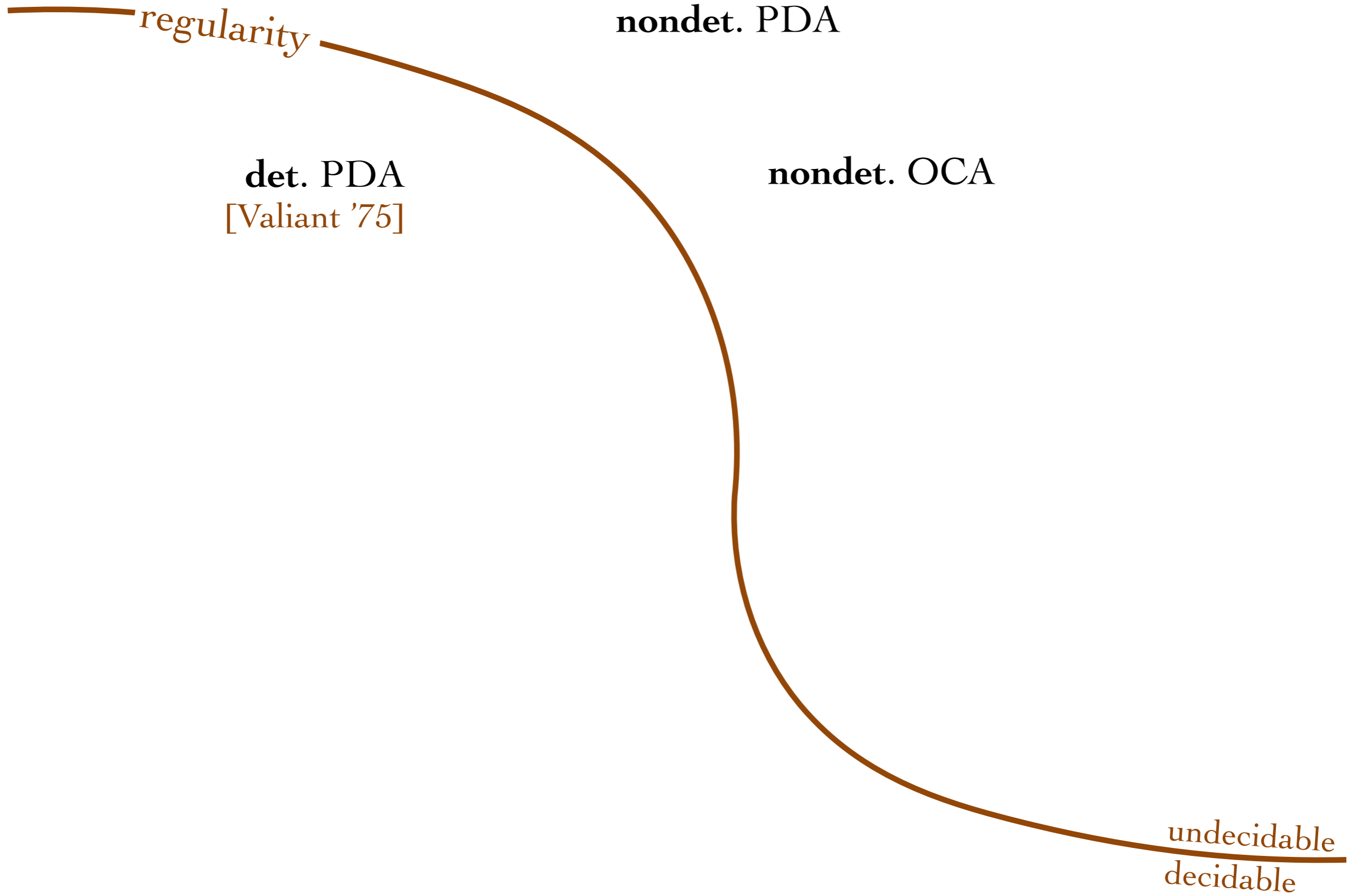Setting parametric in class $C$

regularity

undecidable
decidable

4

regularity

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undecidable
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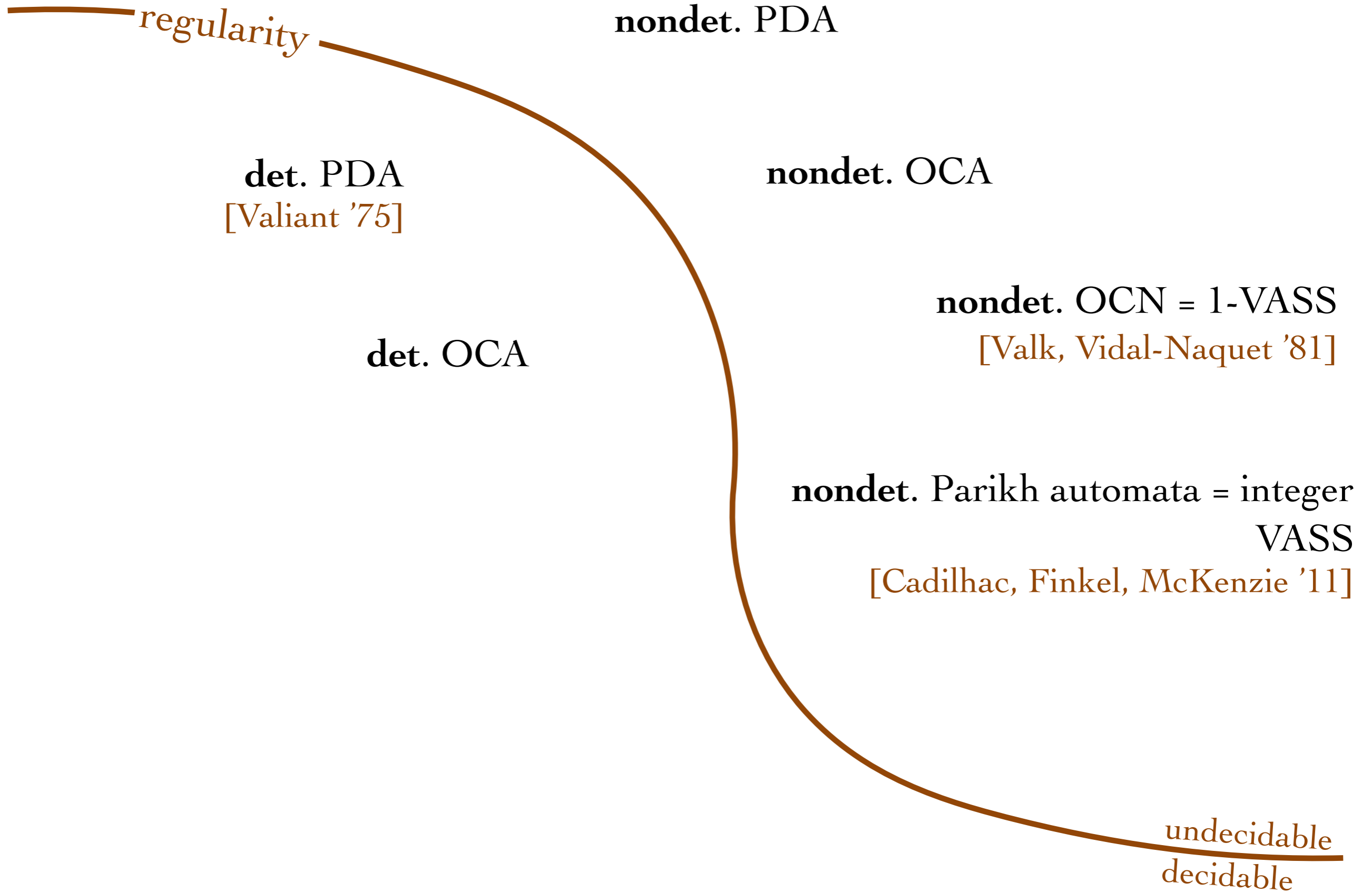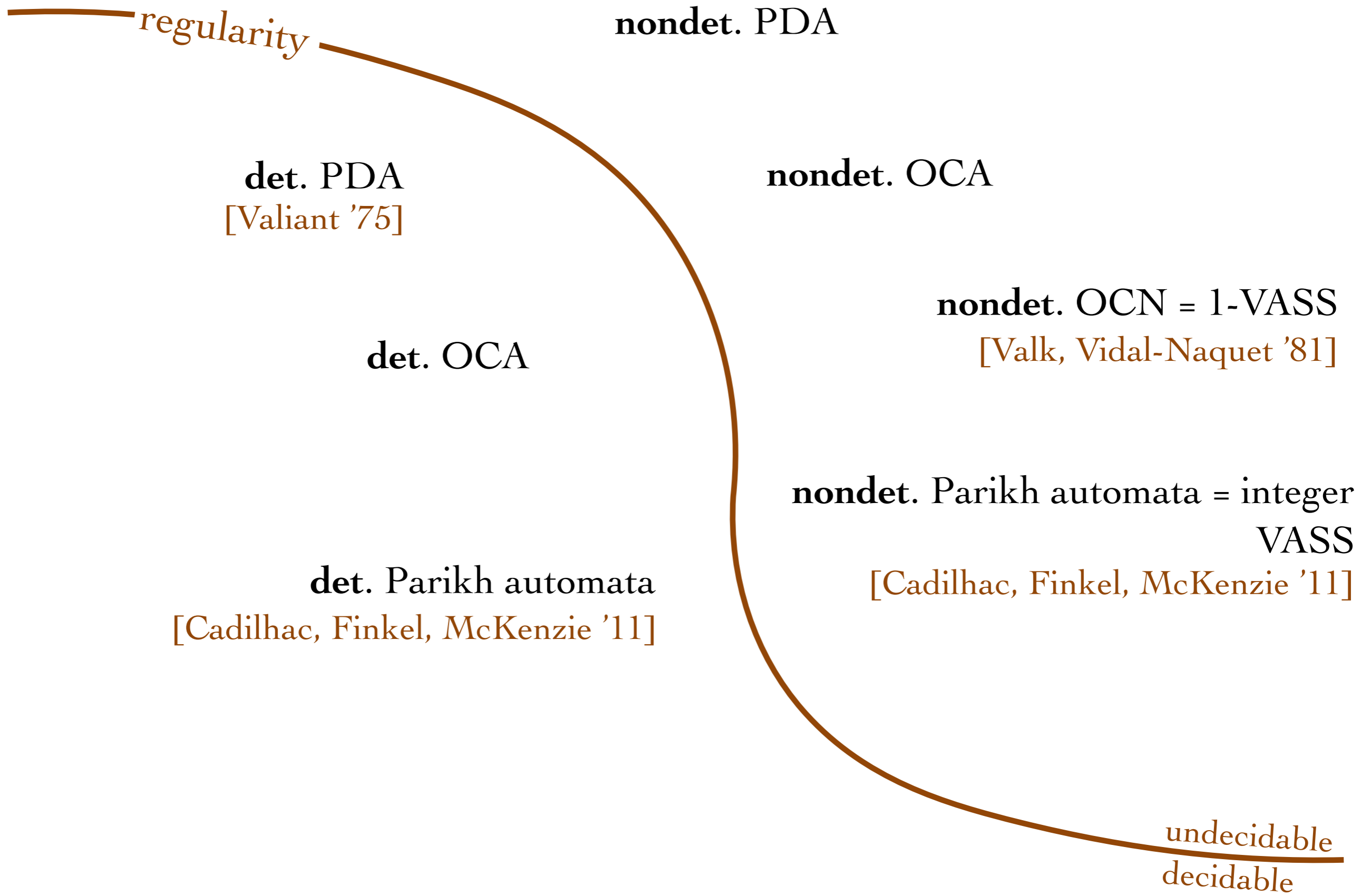
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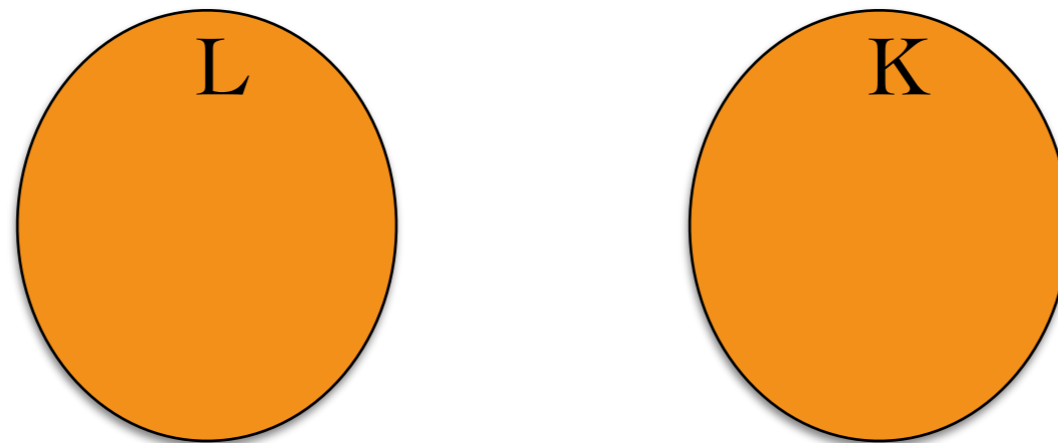
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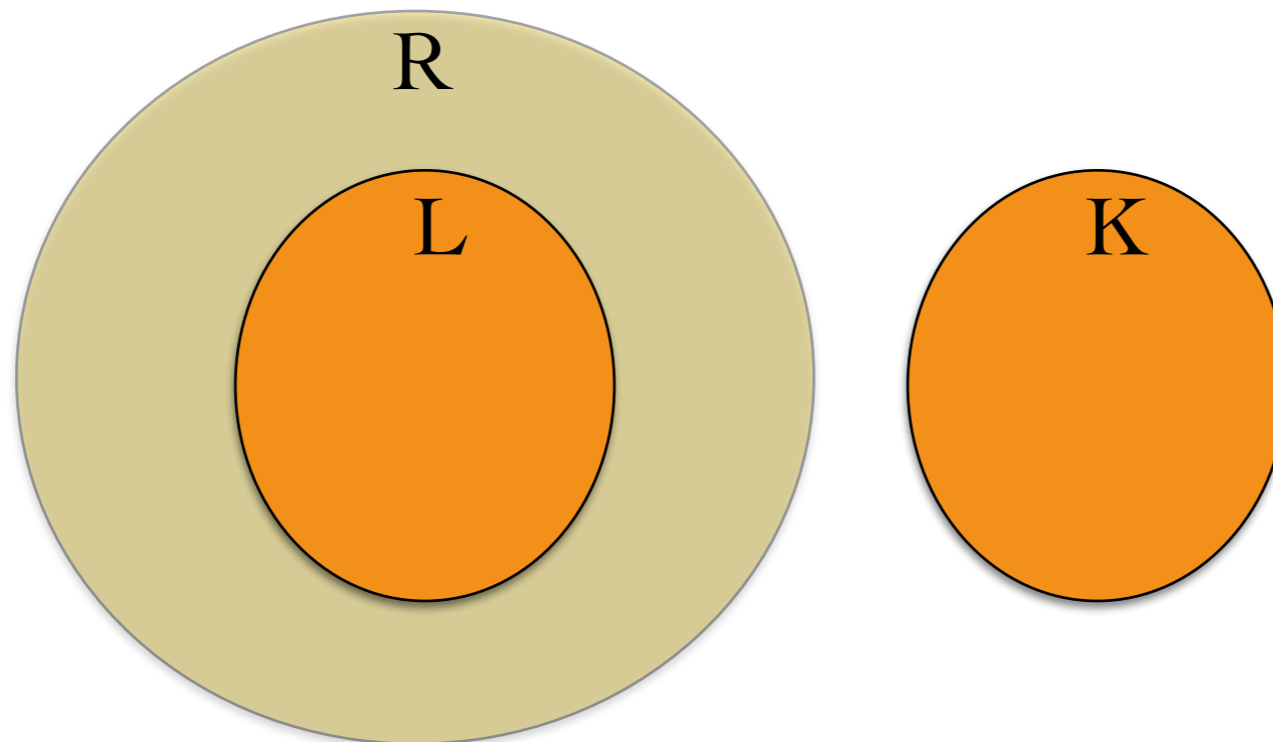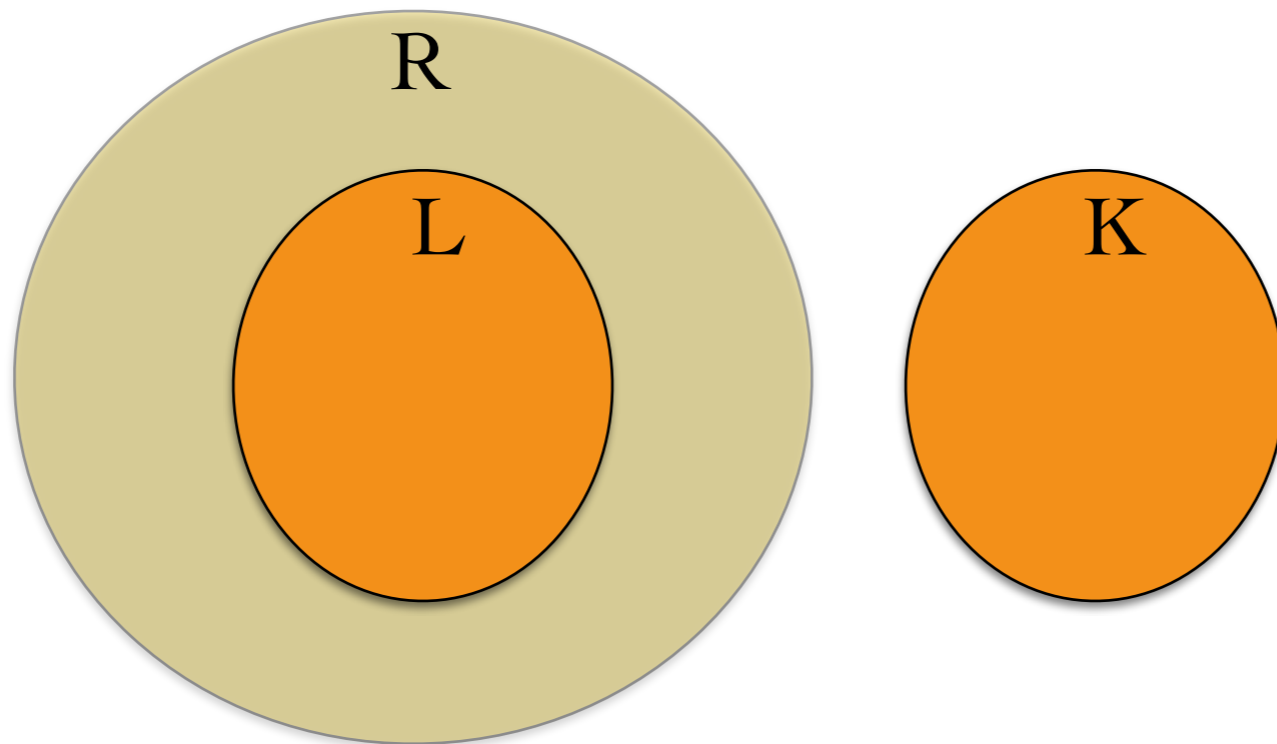


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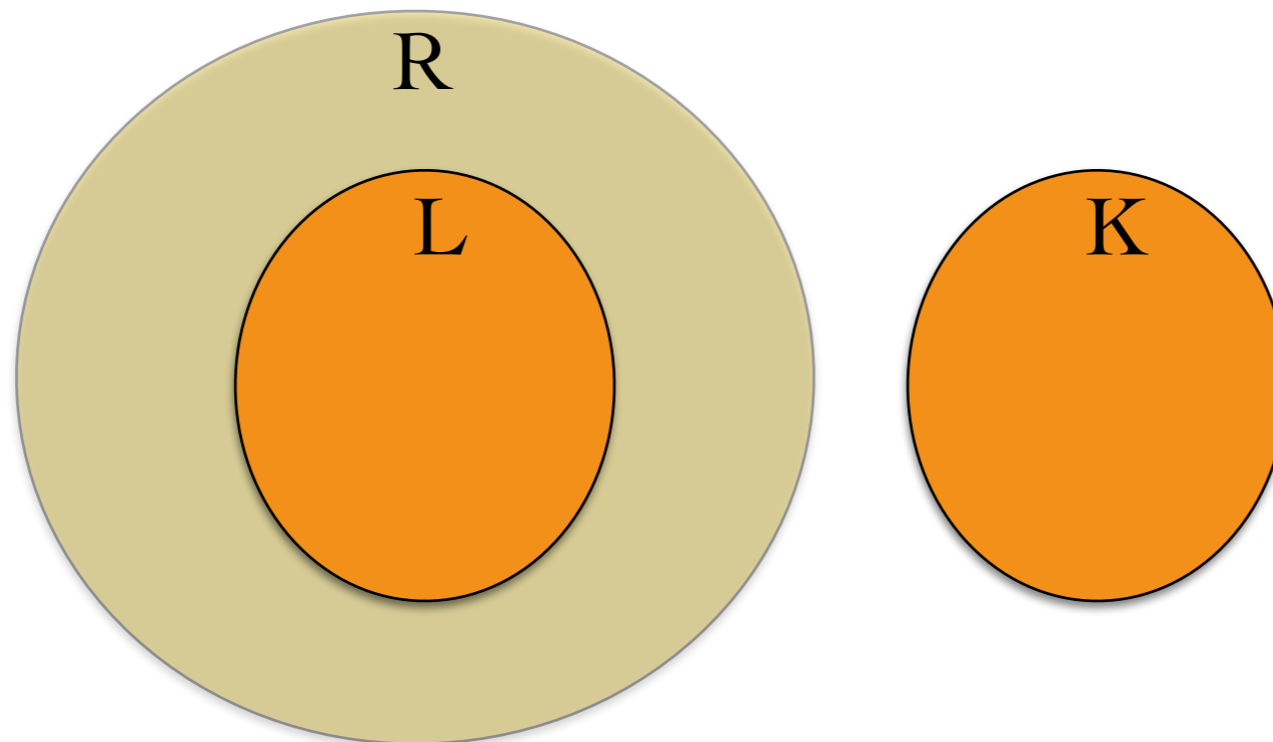


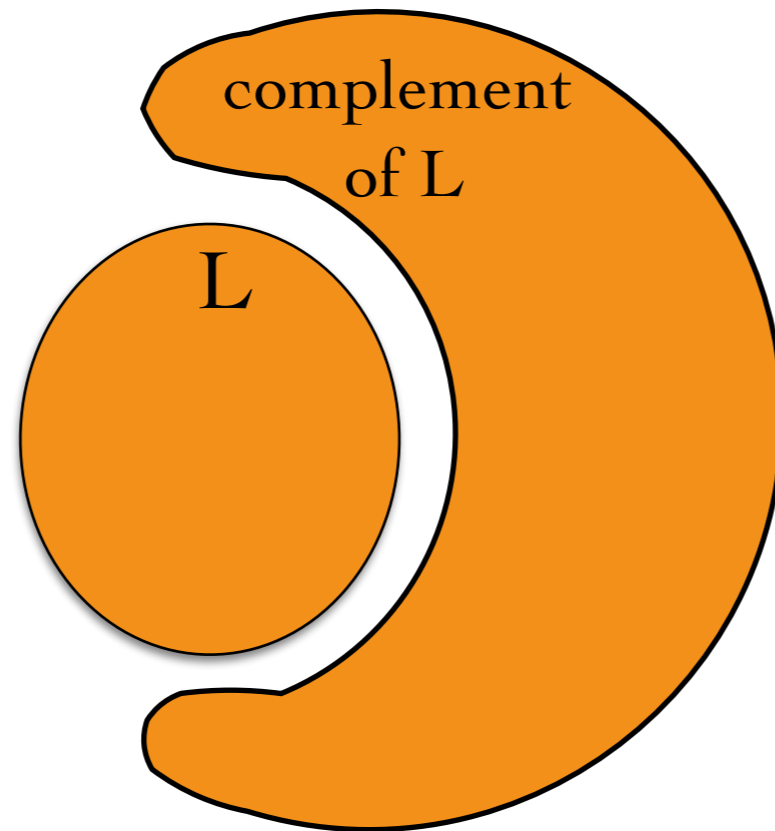Symmetric in L, K

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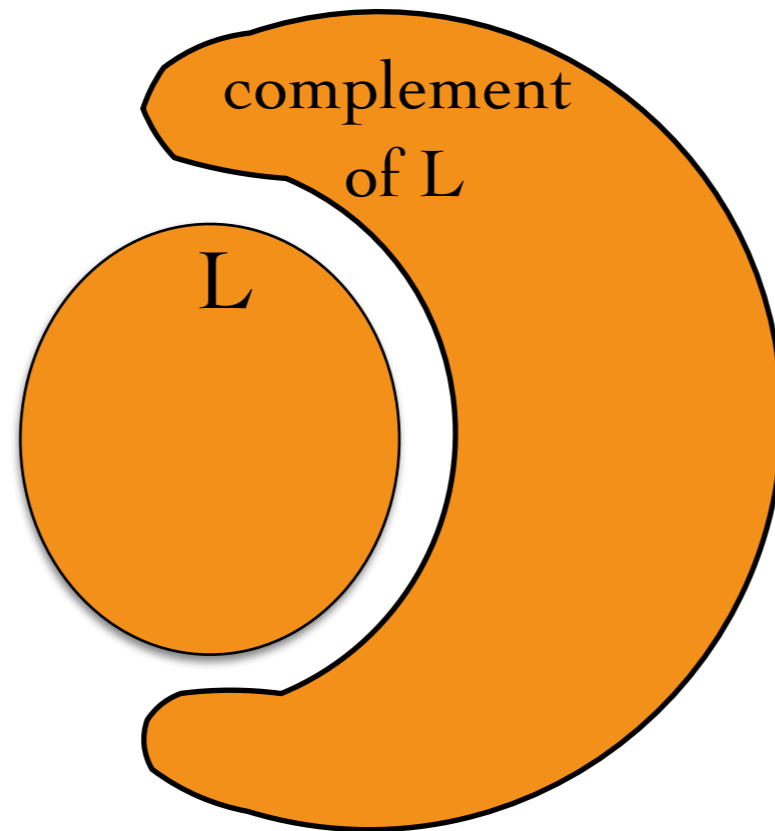


complement of L

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L and its complement are regular separable  iff  L is regular

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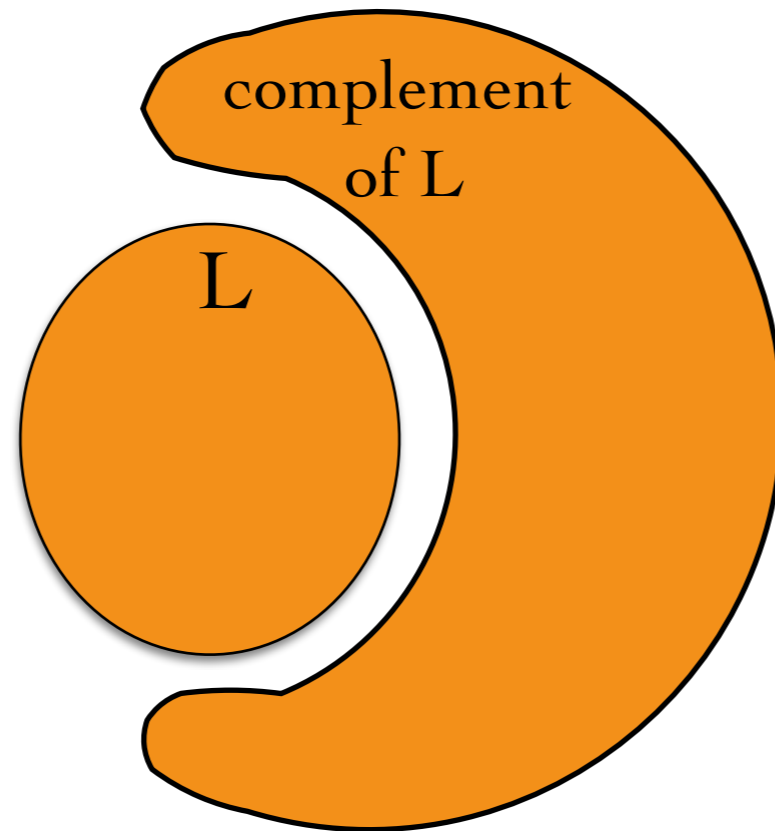
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Regular separability subsumes regularity
whenever *C* is effectively closed under complement

# For regular separability, nondeterminism does not matter
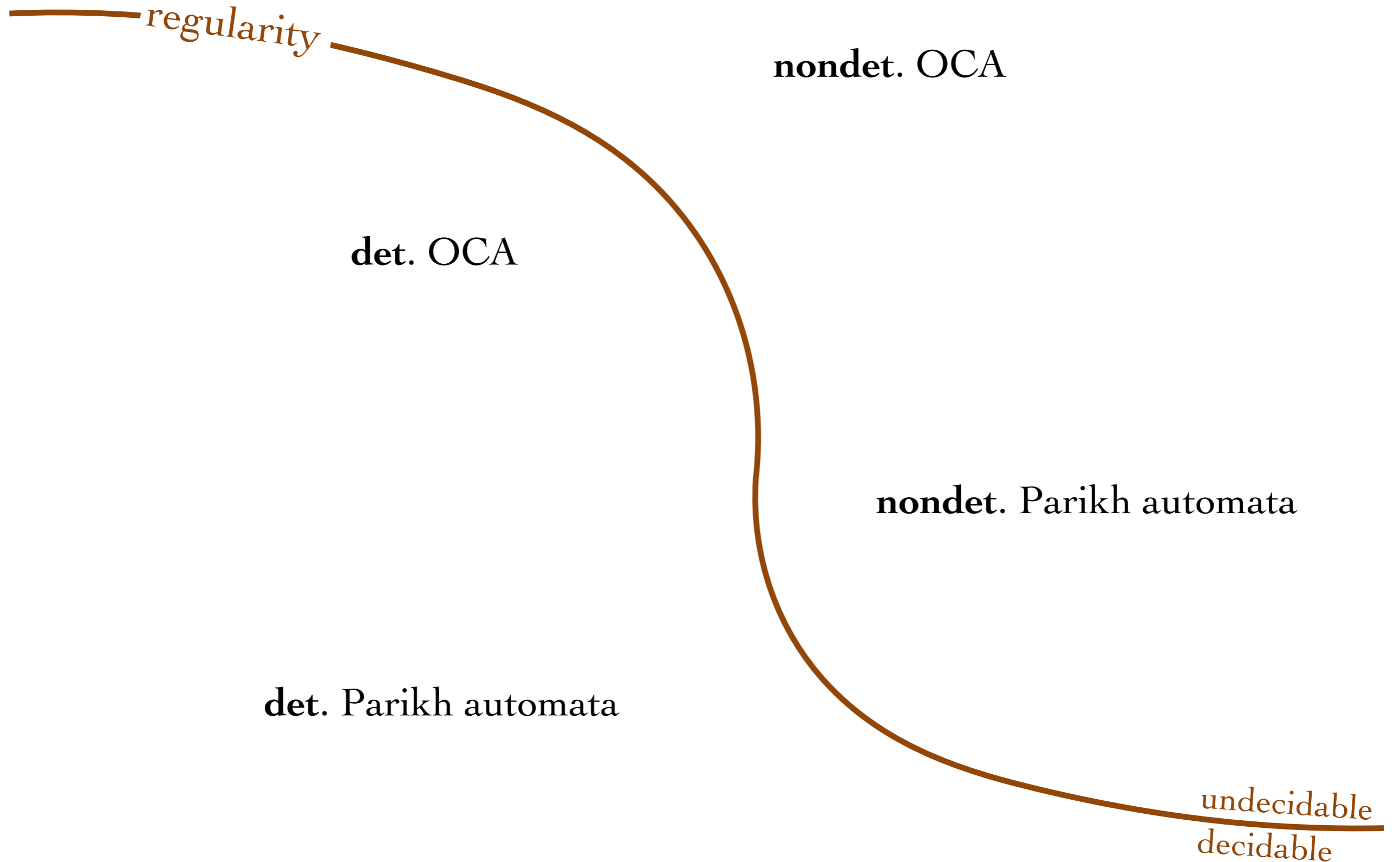
regularity

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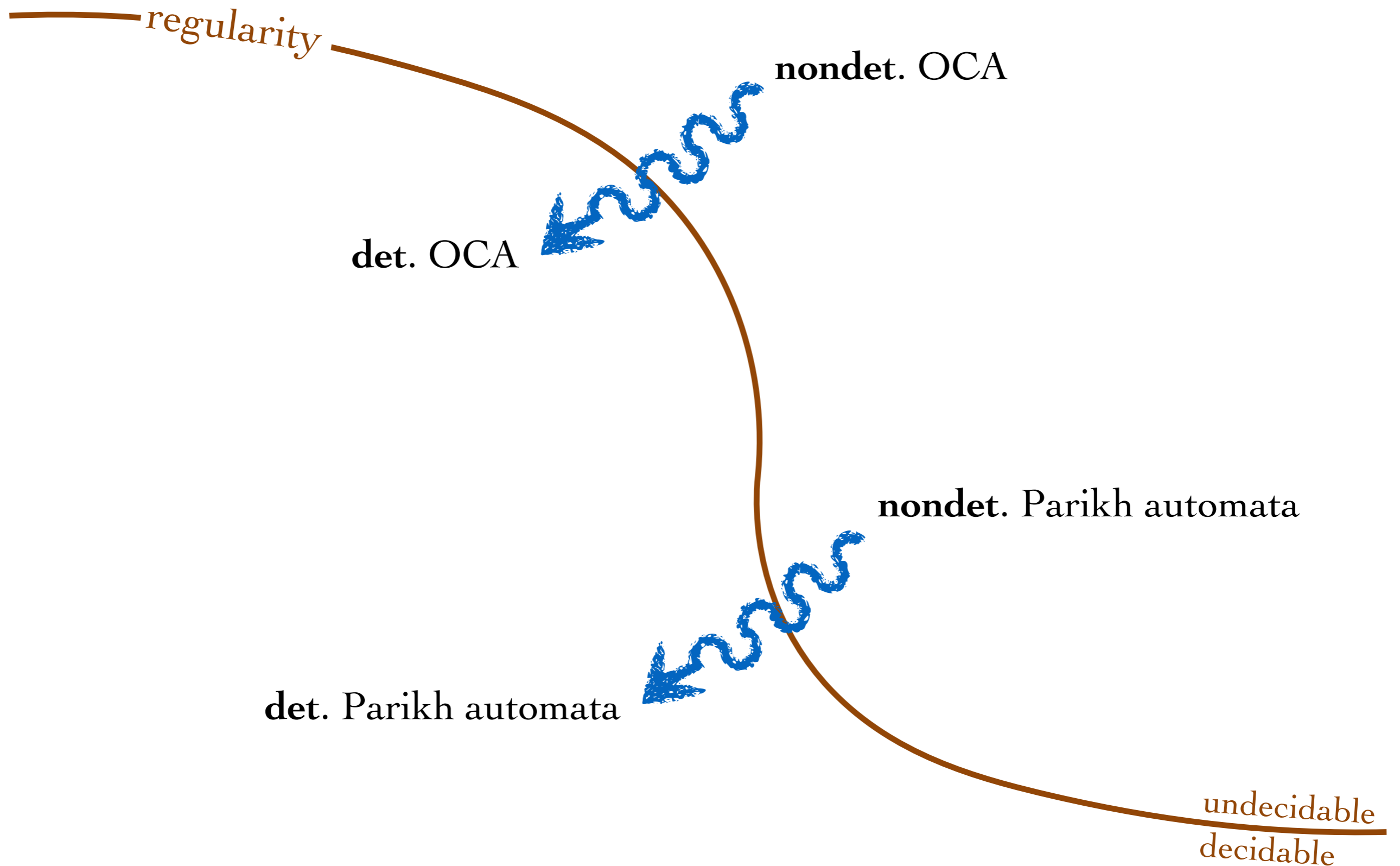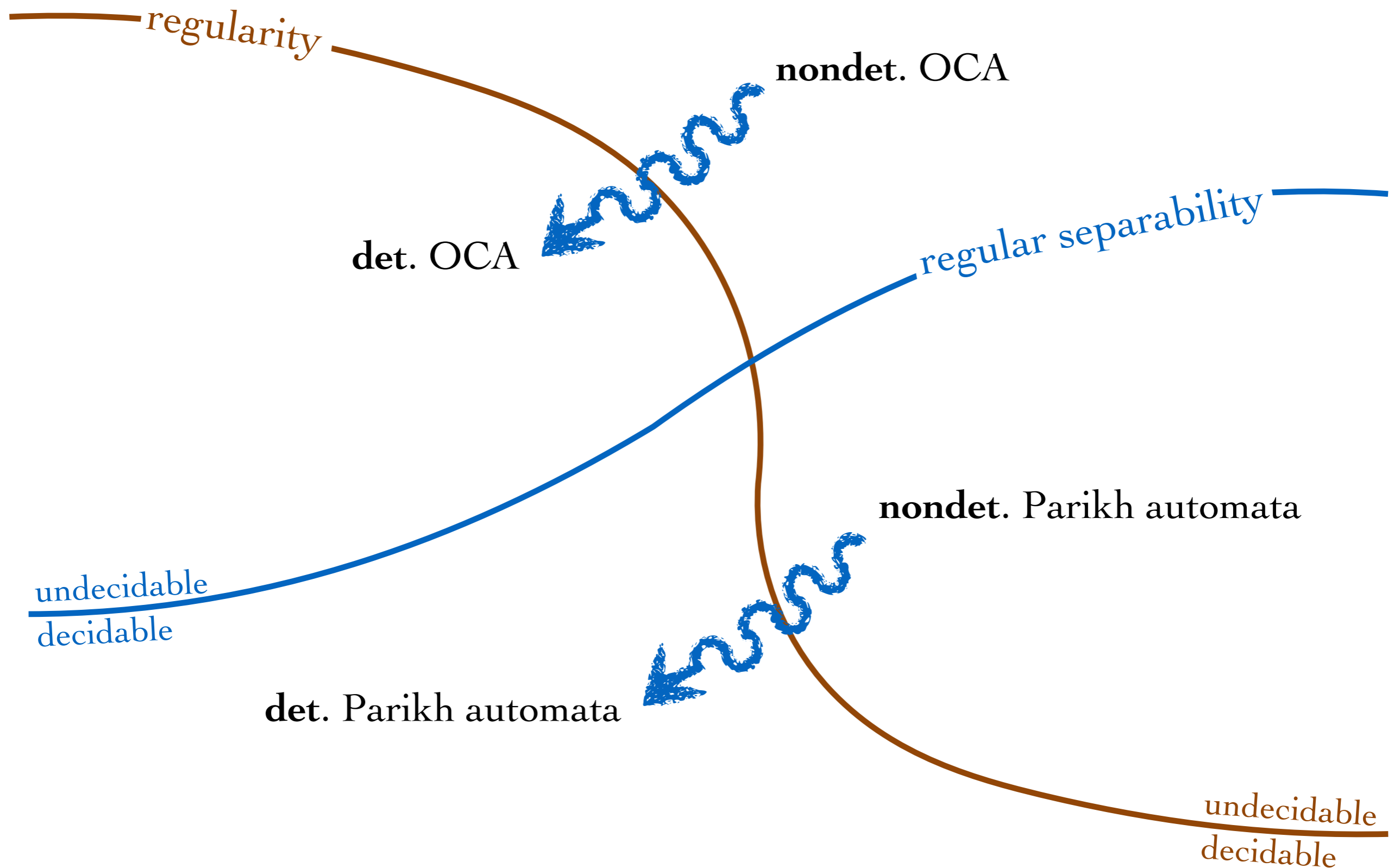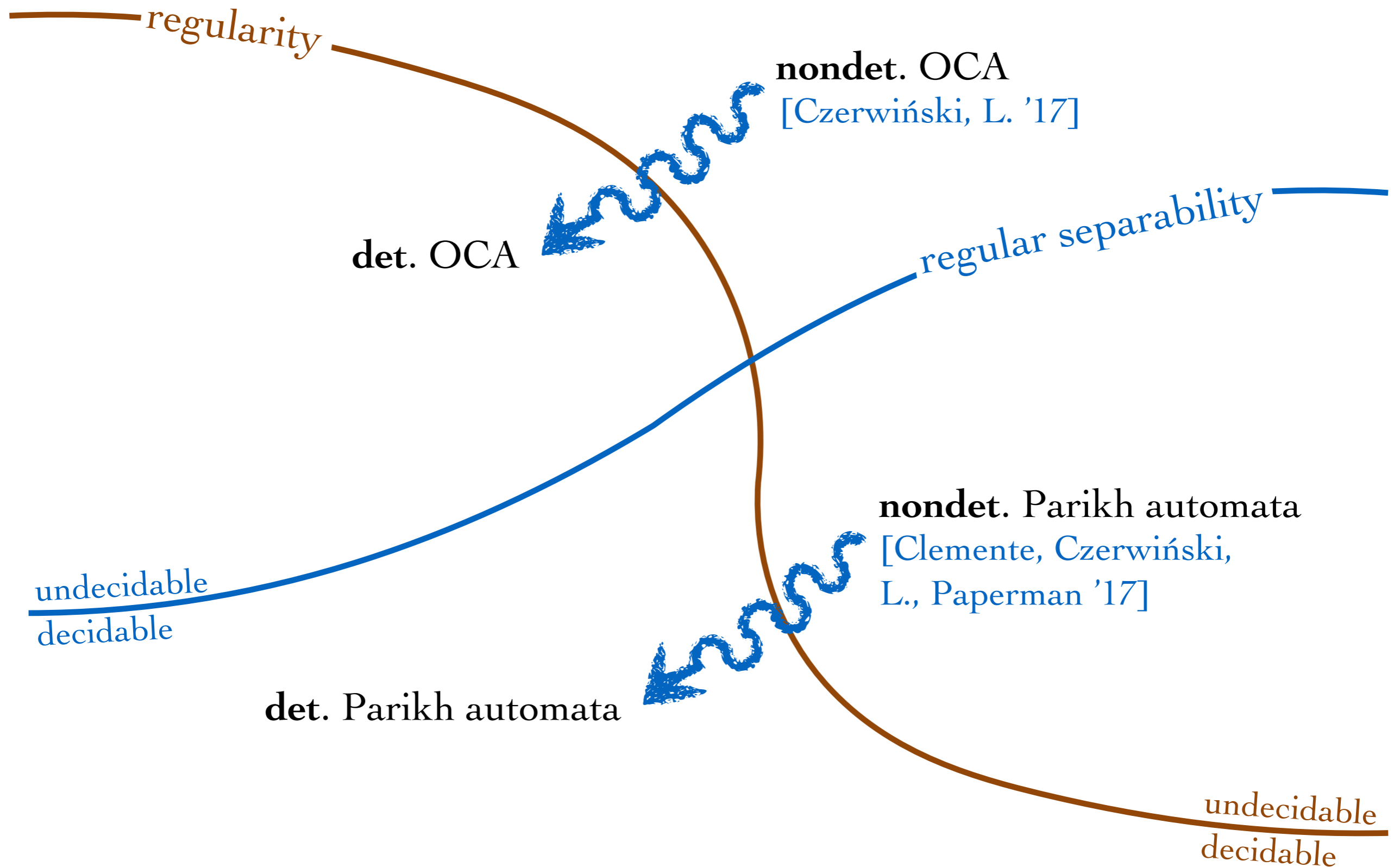
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Then $\ \mathcal{H}(\mathcal{C})$ = languages of **nondeterministic OCA** (effectively)

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**Fact**: both $\mathcal{C}$ and $\mathcal{H}(\mathcal{C})$ are effectively closed under inverse images of homomorphisms from $\mathcal{H}$

Fix a class of languages $C$ and a class of homomorphisms $\mathcal{H}$ s.t. both $C$ and $\mathcal{H}(C)$ are effectively closed under inverse images of $\mathcal{H}$

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L, K in $C$
h, g in $\mathcal{H}$

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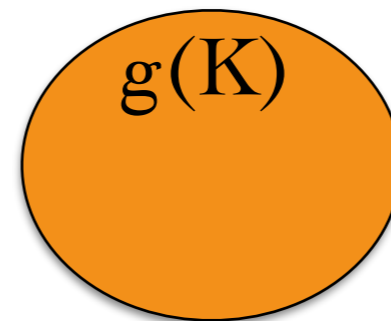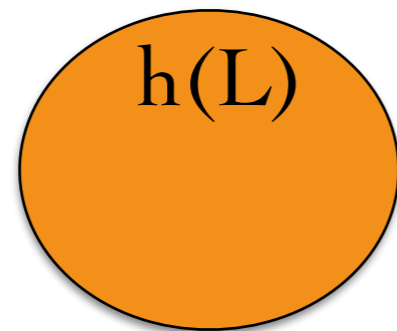Regular separability of $\mathcal{H}(C)$ reduces to regular separability of $C$

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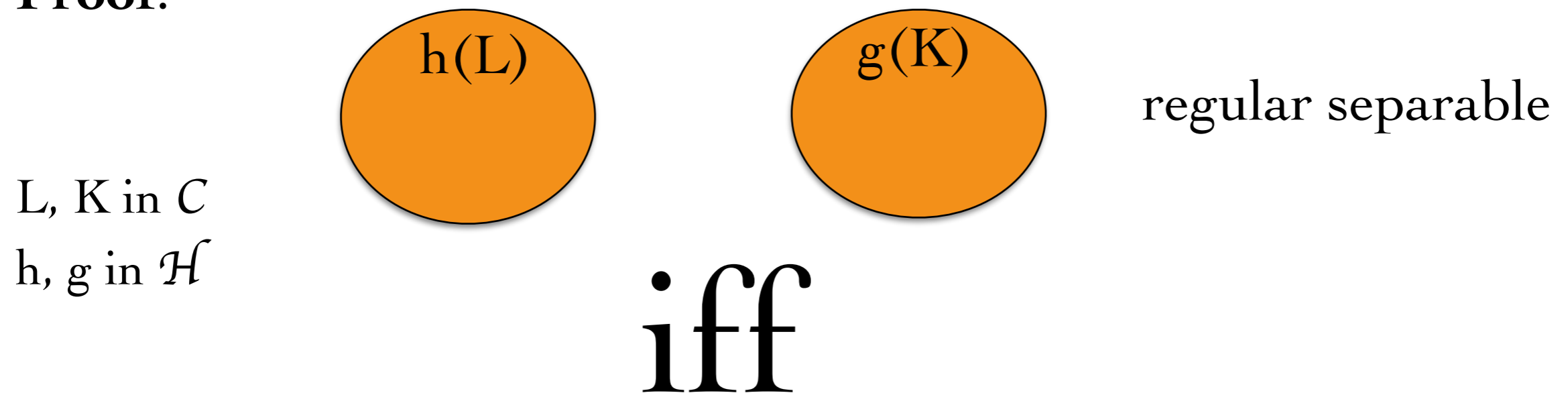
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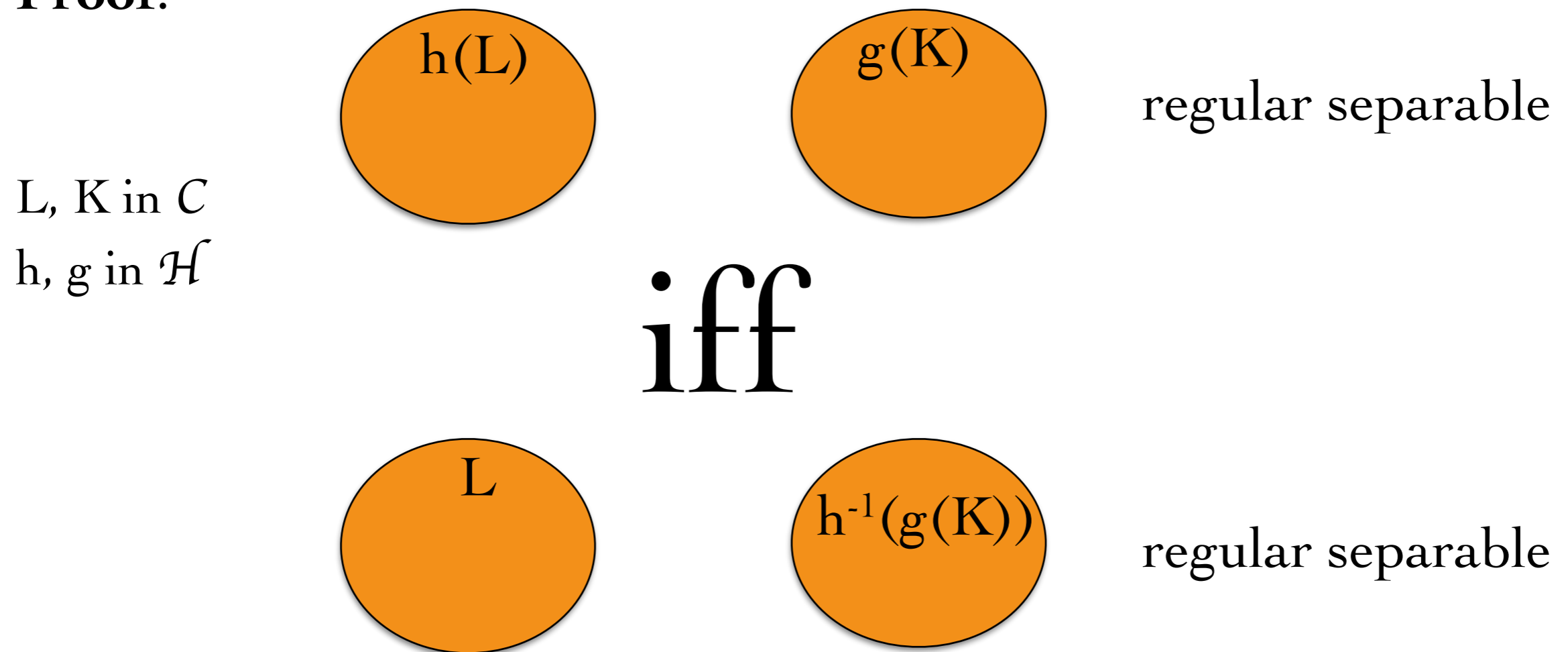
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L    h⁻¹(g(K))           regular separable

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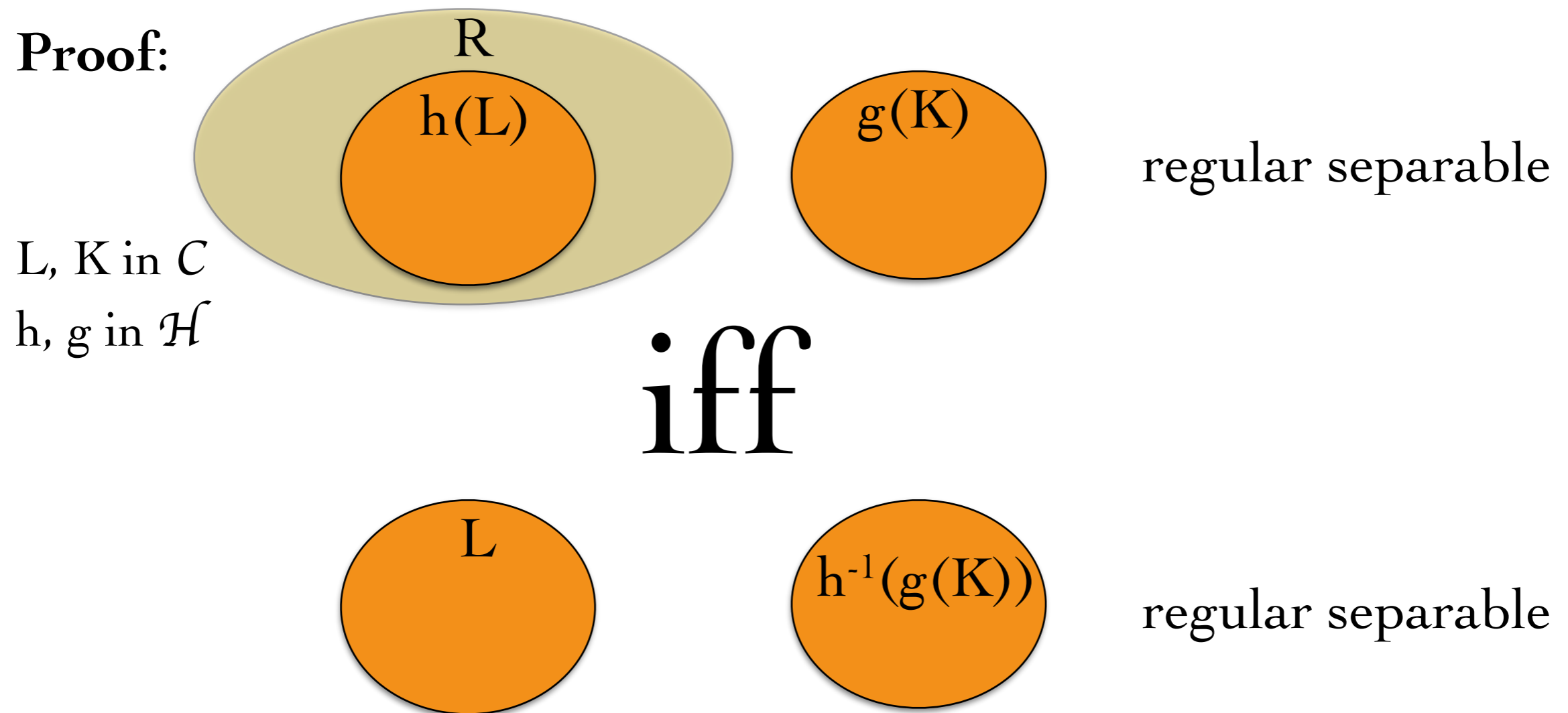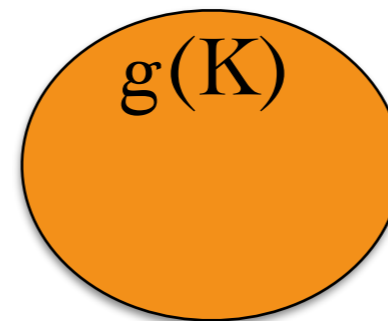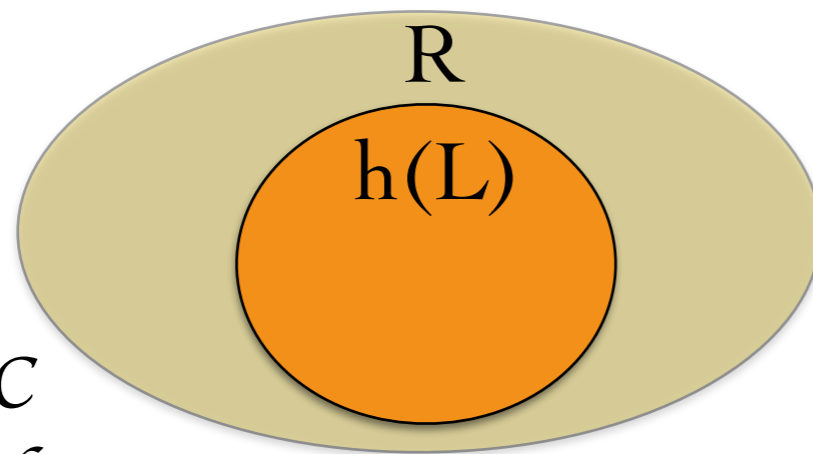regular separable

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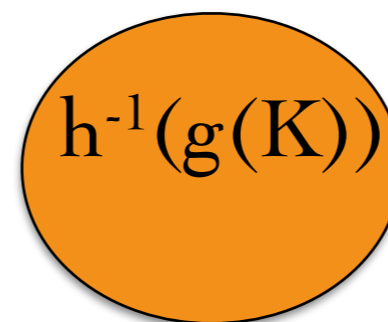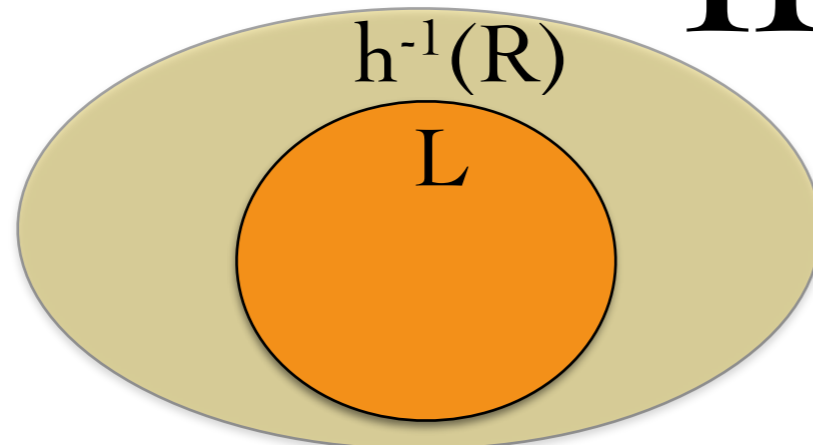Regular separability of $\mathcal{H}(C)$ reduces to regular separability of $C$
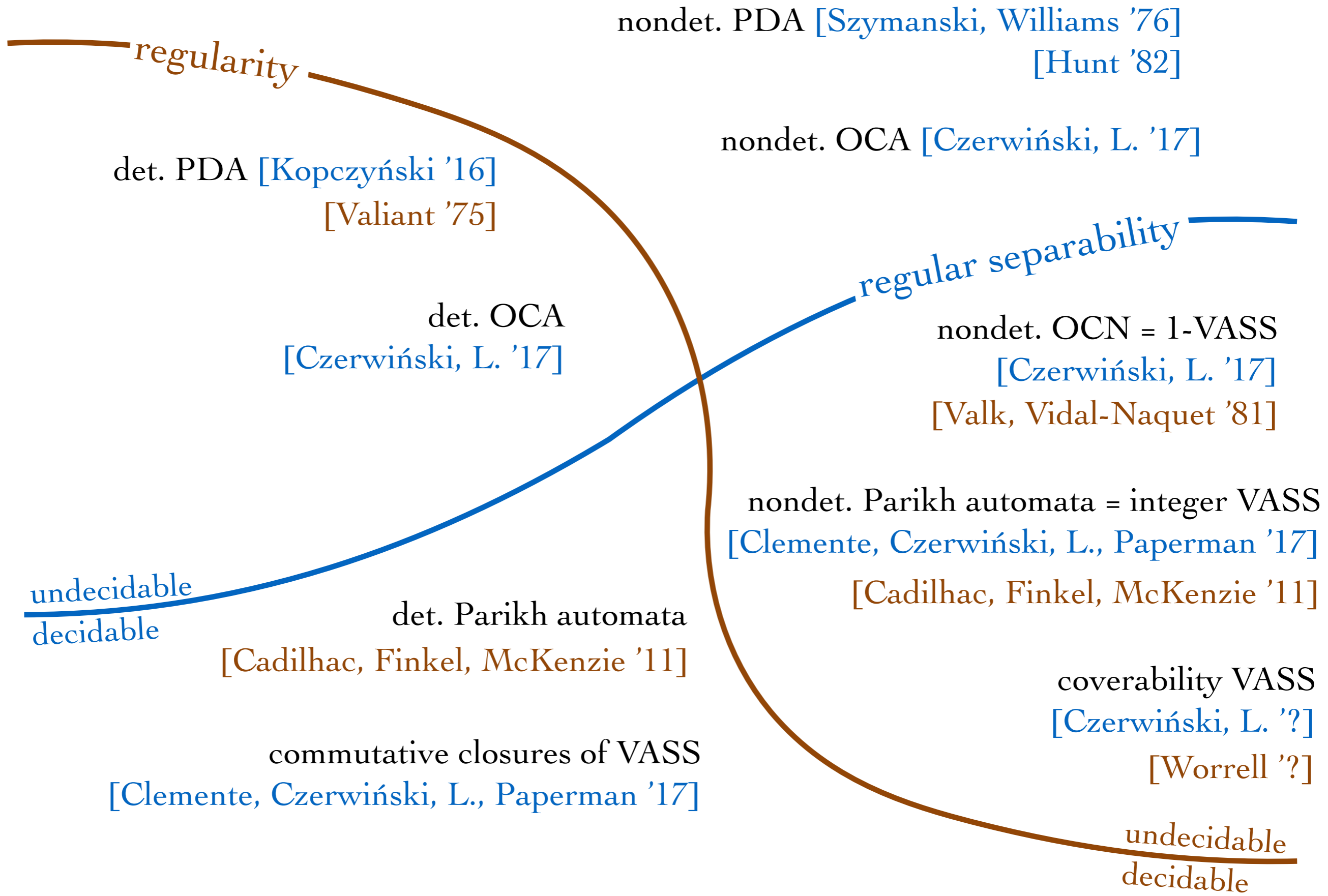
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R

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iff

$h^{-1}(R)$

L

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regular separable

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