

# On Reversible Transducers or Why determinism is outdated (on 2-way transducers)

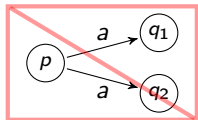
**Luc Dartois, Paulin Fournier, Ismaël Jecker, Nathan Lhote.**

Highlights '17, September 13<sup>th</sup>

A system is called reversible if it can be executed and rewind deterministically.

# Reversibility

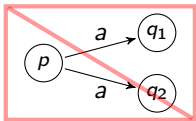
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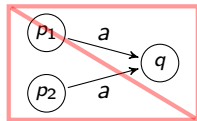
Input deterministic

# Reversibility

A system is called reversible if it can be executed and rewound deterministically.



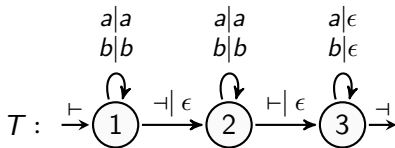
Input deterministic



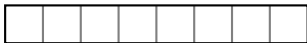
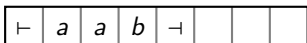
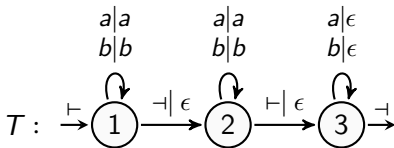
Input co-deterministic

Reversible : input deterministic **and** co-deterministic

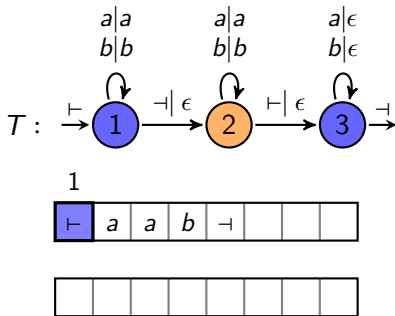
# Two-way transducers



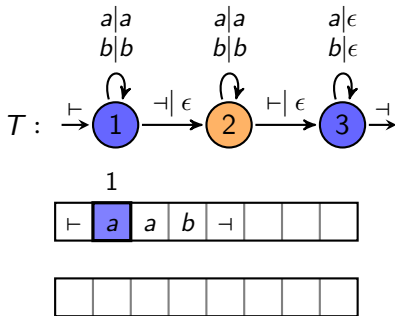
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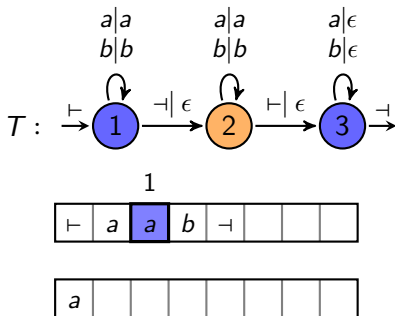


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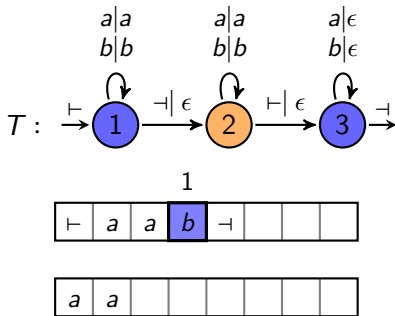




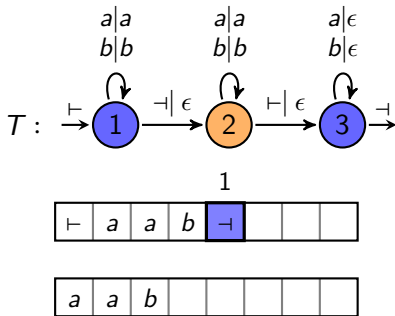
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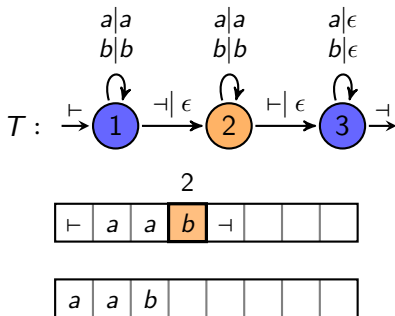
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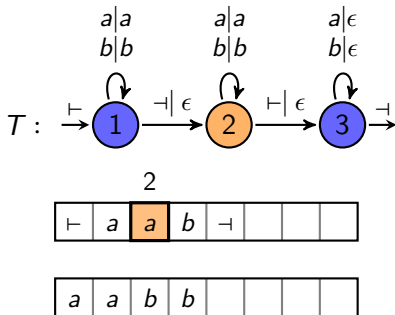
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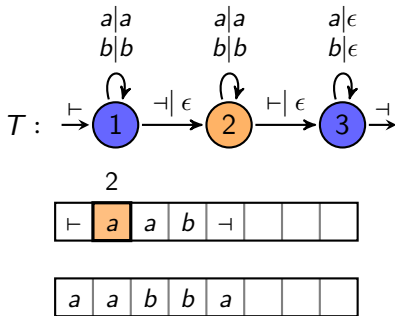
# Two-way transducers



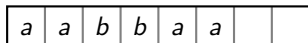
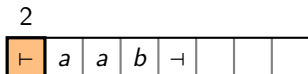
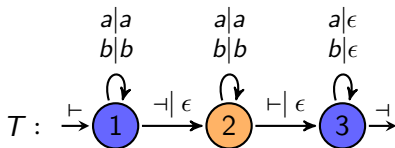
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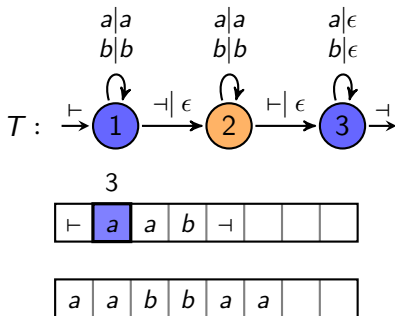
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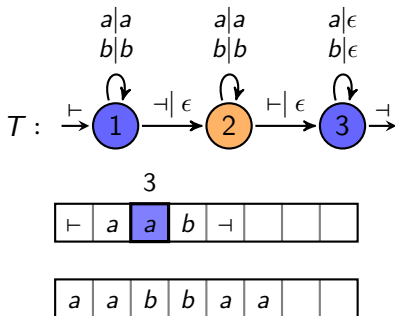


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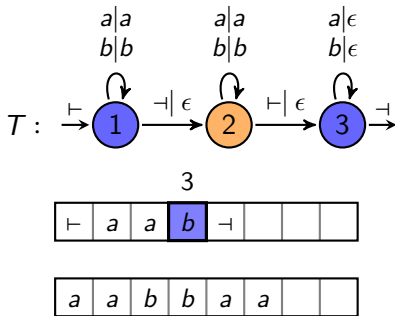




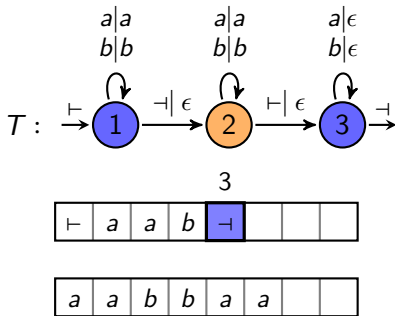
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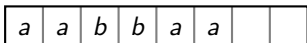
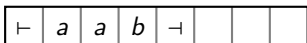
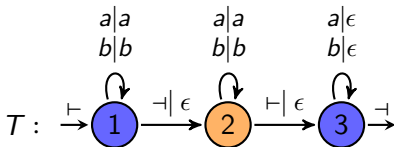
# Two-way transducers



# Two-way transducers



# Two-way transducers



$$\mathcal{F}_T : w \mapsto w\bar{w}$$

# Transducer composition

$$1\text{FT} \circ 1\text{FT} = 1\text{FT}$$

$$2\text{FT} \circ 2\text{FT} = 2\text{FT}$$

$$2\text{FT} \circ 2\text{RFT} = 2\text{FT}$$

$$2\text{RFT} \circ 2\text{RFT} = 2\text{RFT}$$

$$\mathcal{A} \in 1\text{FT} \quad \mathcal{B} \in 1\text{FT}$$

Input tape:  $w$

$a_0$	$a_1$	$a_2$	$a_3$	$a_4$	$a_5$	...
-------	-------	-------	-------	-------	-------	-----

						...
--	--	--	--	--	--	-----

						...
--	--	--	--	--	--	-----

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-------	-------	-------	-------	-------	-------	-----

Working tape:  $\mathcal{A}(w)$

$b_0$	$b_1$	$b_2$	$b_3$	$b_4$	$b_5$	...
-------	-------	-------	-------	-------	-------	-----

						...
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-------	-------	-------	-------	-------	-------	---------

Working tape:  $\mathcal{A}(w)$

$b_0$	$b_1$	$b_2$	$b_3$	$b_4$	$b_5$	$\dots$
-------	-------	-------	-------	-------	-------	---------

Output tape:  $(\mathcal{B} \circ \mathcal{A})(w)$

$c_0$	$c_1$	$c_2$	$c_3$	$c_4$	$c_5$	$\dots$
-------	-------	-------	-------	-------	-------	---------

# Transducer composition

$$1\text{FT} \circ 1\text{FT} = 1\text{FT}$$

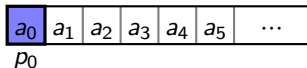
$$2\text{FT} \circ 2\text{FT} = 2\text{FT}$$

$$2\text{FT} \circ 2\text{RFT} = 2\text{FT}$$

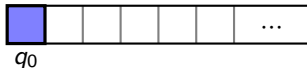
$$2\text{RFT} \circ 2\text{RFT} = 2\text{RFT}$$

$$\mathcal{A} \in 1\text{FT} \quad \mathcal{B} \in 1\text{FT}$$

Input tape:  $w$



Working tape:  $\mathcal{A}(w)$



Output tape:  $(\mathcal{B} \circ \mathcal{A})(w)$





# Transducer composition

$$1\text{FT} \circ 1\text{FT} = 1\text{FT}$$

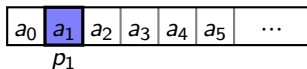
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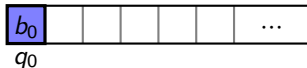
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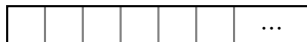
Input tape:  $w$



Working tape:  $\mathcal{A}(w)$



Output tape:  $(\mathcal{B} \circ \mathcal{A})(w)$



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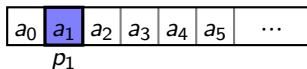
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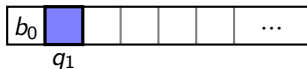
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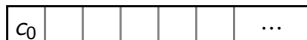
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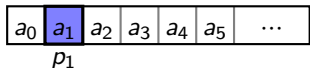
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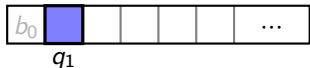
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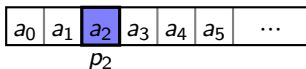
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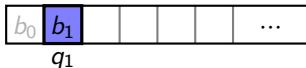
$$2\text{RFT} \circ 2\text{RFT} = 2\text{RFT}$$

$$\mathcal{A} \in 1\text{FT} \quad \mathcal{B} \in 1\text{FT}$$

Input tape:  $w$



Working tape:  $\mathcal{A}(w)$



Output tape:  $(\mathcal{B} \circ \mathcal{A})(w)$



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$$1\text{FT} \circ 1\text{FT} = 1\text{FT}$$

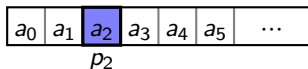
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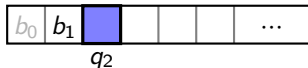
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Working tape:  $\mathcal{A}(w)$



Output tape:  $(\mathcal{B} \circ \mathcal{A})(w)$



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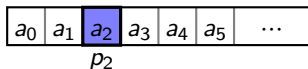
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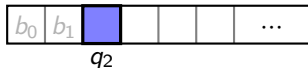
$$2\text{RFT} \circ 2\text{RFT} = 2\text{RFT}$$

$$\mathcal{A} \in 1\text{FT} \quad \mathcal{B} \in 1\text{FT}$$

Input tape:  $w$



Working tape:  $\mathcal{A}(w)$



Output tape:  $(\mathcal{B} \circ \mathcal{A})(w)$



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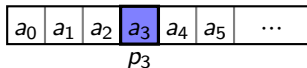
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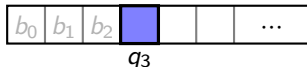
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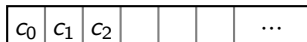
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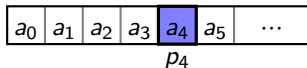
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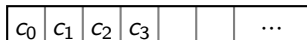
Input tape:  $w$



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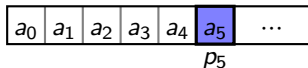
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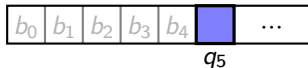
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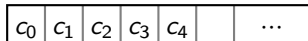
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Input tape:  $w$

$a_0$	$a_1$	$a_2$	$a_3$	$a_4$	$a_5$	$\dots$
-------	-------	-------	-------	-------	-------	---------

Working tape:  $\mathcal{A}(w)$

$b_0$	$b_1$	$b_2$	$b_3$	$b_4$	$b_5$	$\dots$
-------	-------	-------	-------	-------	-------	---------

Output tape:  $(\mathcal{B} \circ \mathcal{A})(w)$

$c_0$	$c_1$	$c_2$	$c_3$	$c_4$	$c_5$	$\dots$
-------	-------	-------	-------	-------	-------	---------

# Transducer composition

$$1\text{FT} \circ 1\text{FT} = 1\text{FT}$$

$$Q_{\mathcal{B} \circ \mathcal{A}} = Q_{\mathcal{A}} \times Q_{\mathcal{B}}$$

$$2\text{FT} \circ 2\text{FT} = 2\text{FT}$$

$$2\text{FT} \circ 2\text{RFT} = 2\text{FT}$$

$$2\text{RFT} \circ 2\text{RFT} = 2\text{RFT}$$

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Working tape:  $\mathcal{A}(w)$

$b_0$	$b_1$	$b_2$	$b_3$	$b_4$	$b_5$	$\dots$
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Output tape:  $(\mathcal{B} \circ \mathcal{A})(w)$

$c_0$	$c_1$	$c_2$	$c_3$	$c_4$	$c_5$	$\dots$
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$$Q_{B \circ A} = Q_A \times Q_B$$

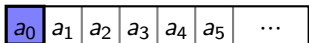
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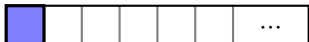
$$\mathcal{A} \in 2\text{FT} \quad \mathcal{B} \in 2\text{FT}$$

Input tape:  $w$



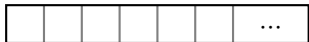
$p_0$

Working tape:  $\mathcal{A}(w)$



$q_0$

Output tape:  $(\mathcal{B} \circ \mathcal{A})(w)$



# Transducer composition

$$1\text{FT} \circ 1\text{FT} = 1\text{FT}$$

$$Q_{B \circ A} = Q_A \times Q_B$$

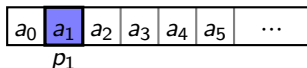
$$2\text{FT} \circ 2\text{FT} = 2\text{FT}$$

$$2\text{FT} \circ 2\text{RFT} = 2\text{FT}$$

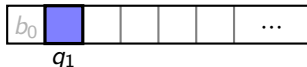
$$2\text{RFT} \circ 2\text{RFT} = 2\text{RFT}$$

$$\mathcal{A} \in 2\text{FT} \quad \mathcal{B} \in 2\text{FT}$$

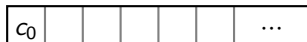
Input tape:  $w$



Working tape:  $\mathcal{A}(w)$



Output tape:  $(\mathcal{B} \circ \mathcal{A})(w)$



# Transducer composition

$$1\text{FT} \circ 1\text{FT} = 1\text{FT}$$

$$Q_{B \circ A} = Q_A \times Q_B$$

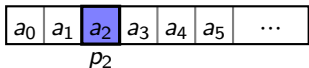
$$2\text{FT} \circ 2\text{FT} = 2\text{FT}$$

$$2\text{FT} \circ 2\text{RFT} = 2\text{FT}$$

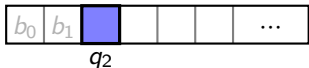
$$2\text{RFT} \circ 2\text{RFT} = 2\text{RFT}$$

$$\mathcal{A} \in 2\text{FT} \quad \mathcal{B} \in 2\text{FT}$$

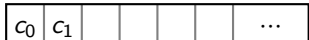
Input tape:  $w$



Working tape:  $\mathcal{A}(w)$



Output tape:  $(\mathcal{B} \circ \mathcal{A})(w)$



# Transducer composition

$$1\text{FT} \circ 1\text{FT} = 1\text{FT}$$

$$Q_{B \circ A} = Q_A \times Q_B$$

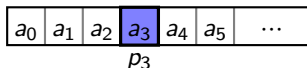
$$2\text{FT} \circ 2\text{FT} = 2\text{FT}$$

$$2\text{FT} \circ 2\text{RFT} = 2\text{FT}$$

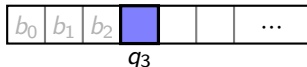
$$2\text{RFT} \circ 2\text{RFT} = 2\text{RFT}$$

$$\mathcal{A} \in 2\text{FT} \quad \mathcal{B} \in 2\text{FT}$$

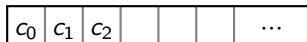
Input tape:  $w$



Working tape:  $\mathcal{A}(w)$



Output tape:  $(\mathcal{B} \circ \mathcal{A})(w)$



# Transducer composition

$$1\text{FT} \circ 1\text{FT} = 1\text{FT}$$

$$Q_{B \circ A} = Q_A \times Q_B$$

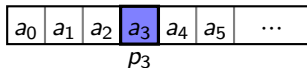
$$2\text{FT} \circ 2\text{FT} = 2\text{FT}$$

$$2\text{FT} \circ 2\text{RFT} = 2\text{FT}$$

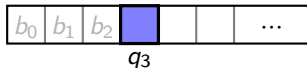
$$2\text{RFT} \circ 2\text{RFT} = 2\text{RFT}$$

$$\mathcal{A} \in 2\text{FT} \quad \mathcal{B} \in 2\text{FT}$$

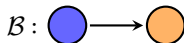
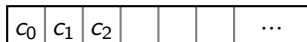
Input tape:  $w$



Working tape:  $\mathcal{A}(w)$



Output tape:  $(\mathcal{B} \circ \mathcal{A})(w)$





# Transducer composition

$$1\text{FT} \circ 1\text{FT} = 1\text{FT}$$

$$Q_{B \circ A} = Q_A \times Q_B$$

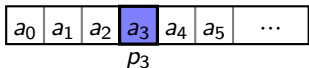
$$2\text{FT} \circ 2\text{FT} = 2\text{FT}$$

$$2\text{FT} \circ 2\text{RFT} = 2\text{FT}$$

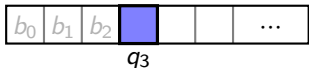
$$2\text{RFT} \circ 2\text{RFT} = 2\text{RFT}$$

$$\mathcal{A} \in 2\text{FT} \quad \mathcal{B} \in 2\text{FT}$$

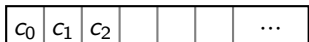
Input tape:  $w$



Working tape:  $\mathcal{A}(w)$



Output tape:  $(\mathcal{B} \circ \mathcal{A})(w)$



# Transducer composition

$$1\text{FT} \circ 1\text{FT} = 1\text{FT}$$

$$Q_{B \circ A} = Q_A \times Q_B$$

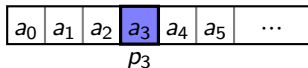
$$2\text{FT} \circ 2\text{FT} = 2\text{FT}$$

$$2\text{FT} \circ 2\text{RFT} = 2\text{FT}$$

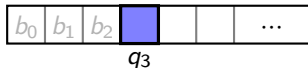
$$2\text{RFT} \circ 2\text{RFT} = 2\text{RFT}$$

$$\mathcal{A} \in 2\text{FT} \quad \mathcal{B} \in 2\text{FT}$$

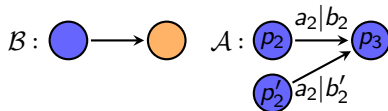
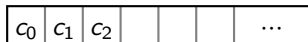
Input tape:  $w$



Working tape:  $\mathcal{A}(w)$



Output tape:  $(\mathcal{B} \circ \mathcal{A})(w)$



# Transducer composition

$$1\text{FT} \circ 1\text{FT} = 1\text{FT}$$

$$Q_{B \circ A} = Q_A \times Q_B$$

$$2\text{FT} \circ 2\text{FT} = 2\text{FT}$$

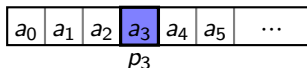
$$Q_{B \circ A} = \eta(Q_A) \times Q_B \text{ [Chytil, Jákł '77]}$$

$$2\text{FT} \circ 2\text{RFT} = 2\text{FT}$$

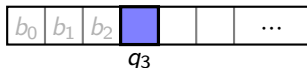
$$2\text{RFT} \circ 2\text{RFT} = 2\text{RFT}$$

$$\mathcal{A} \in 2\text{FT} \quad \mathcal{B} \in 2\text{FT}$$

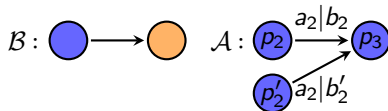
Input tape:  $w$



Working tape:  $\mathcal{A}(w)$



Output tape:  $(\mathcal{B} \circ \mathcal{A})(w)$



# Transducer composition

$$1\text{FT} \circ 1\text{FT} = 1\text{FT}$$

$$Q_{B \circ A} = Q_A \times Q_B$$

$$2\text{FT} \circ 2\text{FT} = 2\text{FT}$$

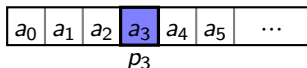
$$Q_{B \circ A} = \eta(Q_A) \times Q_B \quad [\text{Chytil, Ják}l \text{ '77}]$$

$$2\text{FT} \circ 2\text{RFT} = 2\text{FT}$$

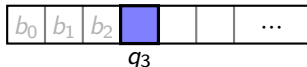
$$2\text{RFT} \circ 2\text{RFT} = 2\text{RFT}$$

$$\mathcal{A} \in 2\text{RFT} \quad \mathcal{B} \in 2\text{FT}$$

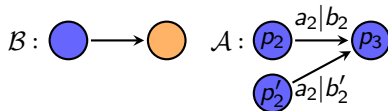
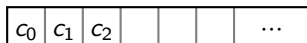
Input tape:  $w$



Working tape:  $\mathcal{A}(w)$



Output tape:  $(\mathcal{B} \circ \mathcal{A})(w)$



# Transducer composition

$$1\text{FT} \circ 1\text{FT} = 1\text{FT}$$

$$Q_{B \circ A} = Q_A \times Q_B$$

$$2\text{FT} \circ 2\text{FT} = 2\text{FT}$$

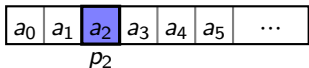
$$Q_{B \circ A} = \eta(Q_A) \times Q_B \quad [\text{Chytil, Ják}l \text{ '77}]$$

$$2\text{FT} \circ 2\text{RFT} = 2\text{FT}$$

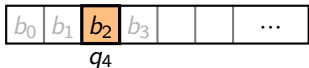
$$2\text{RFT} \circ 2\text{RFT} = 2\text{RFT}$$

$$\mathcal{A} \in 2\text{RFT} \quad \mathcal{B} \in 2\text{FT}$$

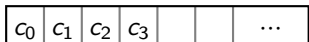
Input tape:  $w$



Working tape:  $\mathcal{A}(w)$



Output tape:  $(\mathcal{B} \circ \mathcal{A})(w)$



# Transducer composition

$$1\text{FT} \circ 1\text{FT} = 1\text{FT}$$

$$Q_{B \circ A} = Q_A \times Q_B$$

$$2\text{FT} \circ 2\text{FT} = 2\text{FT}$$

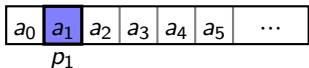
$$Q_{B \circ A} = \eta(Q_A) \times Q_B \quad [\text{Chytil, Ják}l \text{ '77}]$$

$$2\text{FT} \circ 2\text{RFT} = 2\text{FT}$$

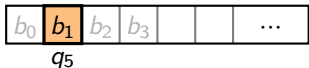
$$2\text{RFT} \circ 2\text{RFT} = 2\text{RFT}$$

$$\mathcal{A} \in 2\text{RFT} \quad \mathcal{B} \in 2\text{FT}$$

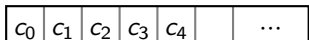
Input tape:  $w$



Working tape:  $\mathcal{A}(w)$



Output tape:  $(\mathcal{B} \circ \mathcal{A})(w)$



# Transducer composition

$$1\text{FT} \circ 1\text{FT} = 1\text{FT}$$

$$Q_{B \circ A} = Q_A \times Q_B$$

$$2\text{FT} \circ 2\text{FT} = 2\text{FT}$$

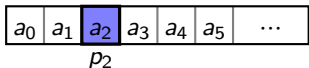
$$Q_{B \circ A} = \eta(Q_A) \times Q_B \quad [\text{Chytil, Ják}l \text{ '77}]$$

$$2\text{FT} \circ 2\text{RFT} = 2\text{FT}$$

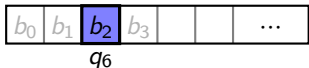
$$2\text{RFT} \circ 2\text{RFT} = 2\text{RFT}$$

$$\mathcal{A} \in 2\text{RFT} \quad \mathcal{B} \in 2\text{FT}$$

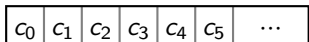
Input tape:  $w$



Working tape:  $\mathcal{A}(w)$



Output tape:  $(\mathcal{B} \circ \mathcal{A})(w)$



# Transducer composition

$$1\text{FT} \circ 1\text{FT} = 1\text{FT}$$

$$Q_{B \circ A} = Q_A \times Q_B$$

$$2\text{FT} \circ 2\text{FT} = 2\text{FT}$$

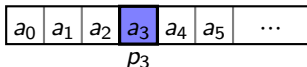
$$Q_{B \circ A} = \eta(Q_A) \times Q_B \quad [\text{Chytil, Ják}l \text{ '77}]$$

$$2\text{FT} \circ 2\text{RFT} = 2\text{FT}$$

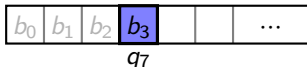
$$2\text{RFT} \circ 2\text{RFT} = 2\text{RFT}$$

$$\mathcal{A} \in 2\text{RFT} \quad \mathcal{B} \in 2\text{FT}$$

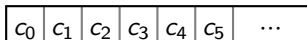
Input tape:  $w$



Working tape:  $\mathcal{A}(w)$



Output tape:  $(\mathcal{B} \circ \mathcal{A})(w)$





# Transducer composition

$$1\text{FT} \circ 1\text{FT} = 1\text{FT}$$

$$Q_{B \circ A} = Q_A \times Q_B$$

$$2\text{FT} \circ 2\text{FT} = 2\text{FT}$$

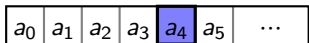
$$Q_{B \circ A} = \eta(Q_A) \times Q_B \quad [\text{Chytil, Ják}l \text{ '77}]$$

$$2\text{FT} \circ 2\text{RFT} = 2\text{FT}$$

$$2\text{RFT} \circ 2\text{RFT} = 2\text{RFT}$$

$$\mathcal{A} \in 2\text{RFT} \quad \mathcal{B} \in 2\text{FT}$$

Input tape:  $w$



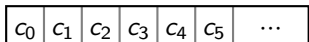
$p_4$

Working tape:  $\mathcal{A}(w)$



$q_8$

Output tape:  $(\mathcal{B} \circ \mathcal{A})(w)$



# Transducer composition

$$1\text{FT} \circ 1\text{FT} = 1\text{FT}$$

$$Q_{B \circ A} = Q_A \times Q_B$$

$$2\text{FT} \circ 2\text{FT} = 2\text{FT}$$

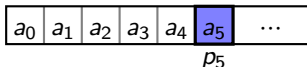
$$Q_{B \circ A} = \eta(Q_A) \times Q_B \quad [\text{Chytil, Ják}l \text{ '77}]$$

$$2\text{FT} \circ 2\text{RFT} = 2\text{FT}$$

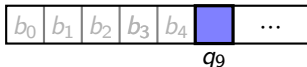
$$2\text{RFT} \circ 2\text{RFT} = 2\text{RFT}$$

$$\mathcal{A} \in 2\text{RFT} \quad \mathcal{B} \in 2\text{FT}$$

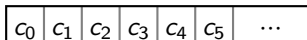
Input tape:  $w$



Working tape:  $\mathcal{A}(w)$



Output tape:  $(\mathcal{B} \circ \mathcal{A})(w)$



# Transducer composition

$$1\text{FT} \circ 1\text{FT} = 1\text{FT}$$

$$Q_{B \circ A} = Q_A \times Q_B$$

$$2\text{FT} \circ 2\text{FT} = 2\text{FT}$$

$$Q_{B \circ A} = \eta(Q_A) \times Q_B \quad [\text{Chytil, Ják}l \text{ '77}]$$

$$2\text{FT} \circ 2\text{RFT} = 2\text{FT}$$

$$2\text{RFT} \circ 2\text{RFT} = 2\text{RFT}$$

$$\mathcal{A} \in 2\text{RFT} \quad \mathcal{B} \in 2\text{FT}$$

Input tape:  $w$

$a_0$	$a_1$	$a_2$	$a_3$	$a_4$	$a_5$	$\dots$
-------	-------	-------	-------	-------	-------	---------

Working tape:  $\mathcal{A}(w)$

$b_0$	$b_1$	$b_2$	$b_3$	$b_4$	$b_5$	$\dots$
-------	-------	-------	-------	-------	-------	---------

Output tape:  $(\mathcal{B} \circ \mathcal{A})(w)$

$c_0$	$c_1$	$c_2$	$c_3$	$c_4$	$c_5$	$\dots$
-------	-------	-------	-------	-------	-------	---------

# Transducer composition

$$1\text{FT} \circ 1\text{FT} = 1\text{FT}$$

$$Q_{B \circ A} = Q_A \times Q_B$$

$$2\text{FT} \circ 2\text{FT} = 2\text{FT}$$

$$Q_{B \circ A} = \eta(Q_A) \times Q_B \quad [\text{Chytil, Ják}l \text{ '77}]$$

$$2\text{FT} \circ 2\text{RFT} = 2\text{FT}$$

$$Q_{B \circ A} = Q_A \times Q_B$$

$$2\text{RFT} \circ 2\text{RFT} = 2\text{RFT}$$

$$\mathcal{A} \in 2\text{RFT} \quad \mathcal{B} \in 2\text{FT}$$

Input tape:  $w$

$a_0$	$a_1$	$a_2$	$a_3$	$a_4$	$a_5$	$\dots$
-------	-------	-------	-------	-------	-------	---------

Working tape:  $\mathcal{A}(w)$

$b_0$	$b_1$	$b_2$	$b_3$	$b_4$	$b_5$	$\dots$
-------	-------	-------	-------	-------	-------	---------

Output tape:  $(\mathcal{B} \circ \mathcal{A})(w)$

$c_0$	$c_1$	$c_2$	$c_3$	$c_4$	$c_5$	$\dots$
-------	-------	-------	-------	-------	-------	---------

# Transducer composition

$$1\text{FT} \circ 1\text{FT} = 1\text{FT}$$

$$Q_{B \circ A} = Q_A \times Q_B$$

$$2\text{FT} \circ 2\text{FT} = 2\text{FT}$$

$$Q_{B \circ A} = \eta(Q_A) \times Q_B \quad [\text{Chytil, Ják}l \text{ '77}]$$

$$2\text{FT} \circ 2\text{RFT} = 2\text{FT}$$

$$Q_{B \circ A} = Q_A \times Q_B$$

$$2\text{RFT} \circ 2\text{RFT} = 2\text{RFT}$$

$$Q_{B \circ A} = Q_A \times Q_B$$

$$\mathcal{A} \in 2\text{RFT} \quad \mathcal{B} \in 2\text{RFT}$$

Input tape:  $w$

$a_0$	$a_1$	$a_2$	$a_3$	$a_4$	$a_5$	$\dots$
-------	-------	-------	-------	-------	-------	---------

Working tape:  $\mathcal{A}(w)$

$b_0$	$b_1$	$b_2$	$b_3$	$b_4$	$b_5$	$\dots$
-------	-------	-------	-------	-------	-------	---------

Output tape:  $(\mathcal{B} \circ \mathcal{A})(w)$

$c_0$	$c_1$	$c_2$	$c_3$	$c_4$	$c_5$	$\dots$
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Reversible Two-way Transducers are as expressive as functional Two-way Transducers !

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## Main Theorem

EXP

Let  $T \in 2FT$ . Then there exists  $T' \in 2RFT$  such that  $T' = T$ .

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## Theorem [Rodrigo de Souza, 2013]

4-EXP

Let  $T \in 2FT$ . Then there exists  $T' \in 2DFT$  such that  $T' = T$ .



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Other Results:

## Theorem [Hopcroft, Ullman. 1967]

EXP

Let  $T_1 \in 1DFT$ ,  $T_2 \in 2FT$ . Then  $\exists T' \in 2FT$  s.t.  $T' = T_2 \circ T_1$ .

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EXP  $\rightarrow$  POLY

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# Expressivity

Reversible Two-way Transducers are as expressive as functional Two-way Transducers !

## Main Theorem

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Other Results:

## Theorem

EXP  $\rightarrow$  POLY

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## Theorem

POLY

Let  $S \in SST$ . Then there exists  $T' \in 2RFT$  such that  $T' = S$ .