Recursion Schemes and the WMSO+U Logic

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Higher-order recursion schemes – what is this?

Definition

Recursion schemes = simply-typed lambda-calculus + recursion

In other words:

- programs with recursion
- higher-order functions (i.e., functions taking other functions as parameters)
- every function/parameter has a fixed type
- no data values, only functions

Higher-order recursion schemes – example

```
fun f(x) {
    a(x);
    if * then f(x);
    b(x);
}
f(x)
```

branching (we are not sure what will be chosen)

recursion

uninterpreted constants (unknown functions)

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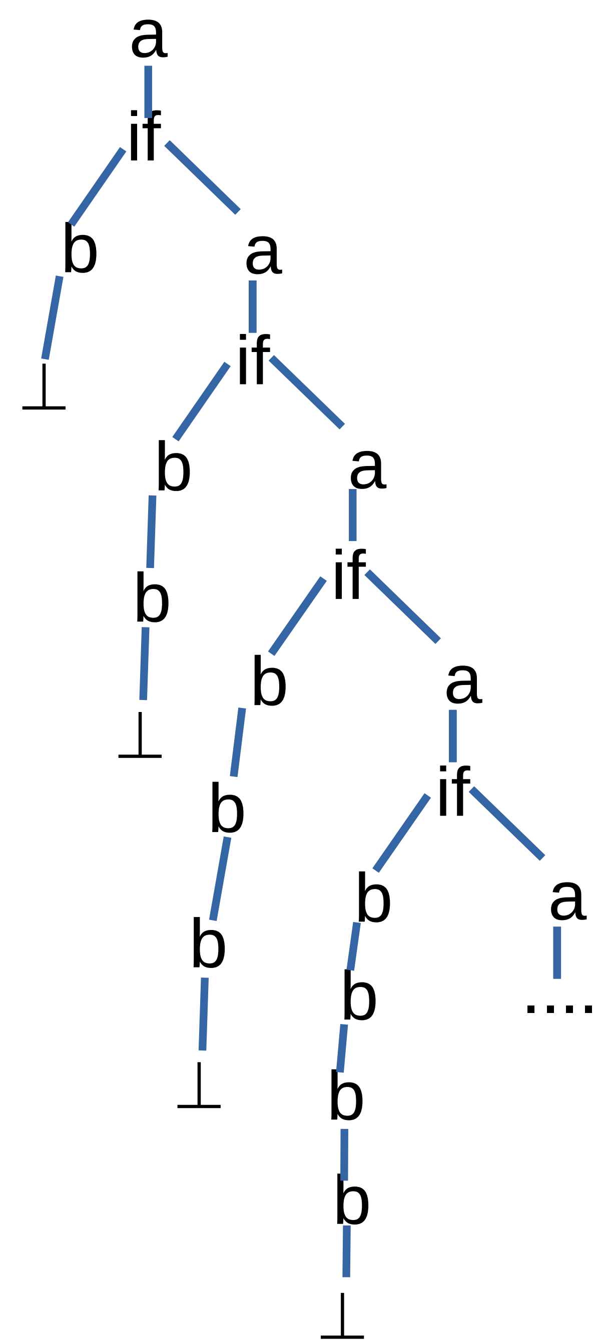
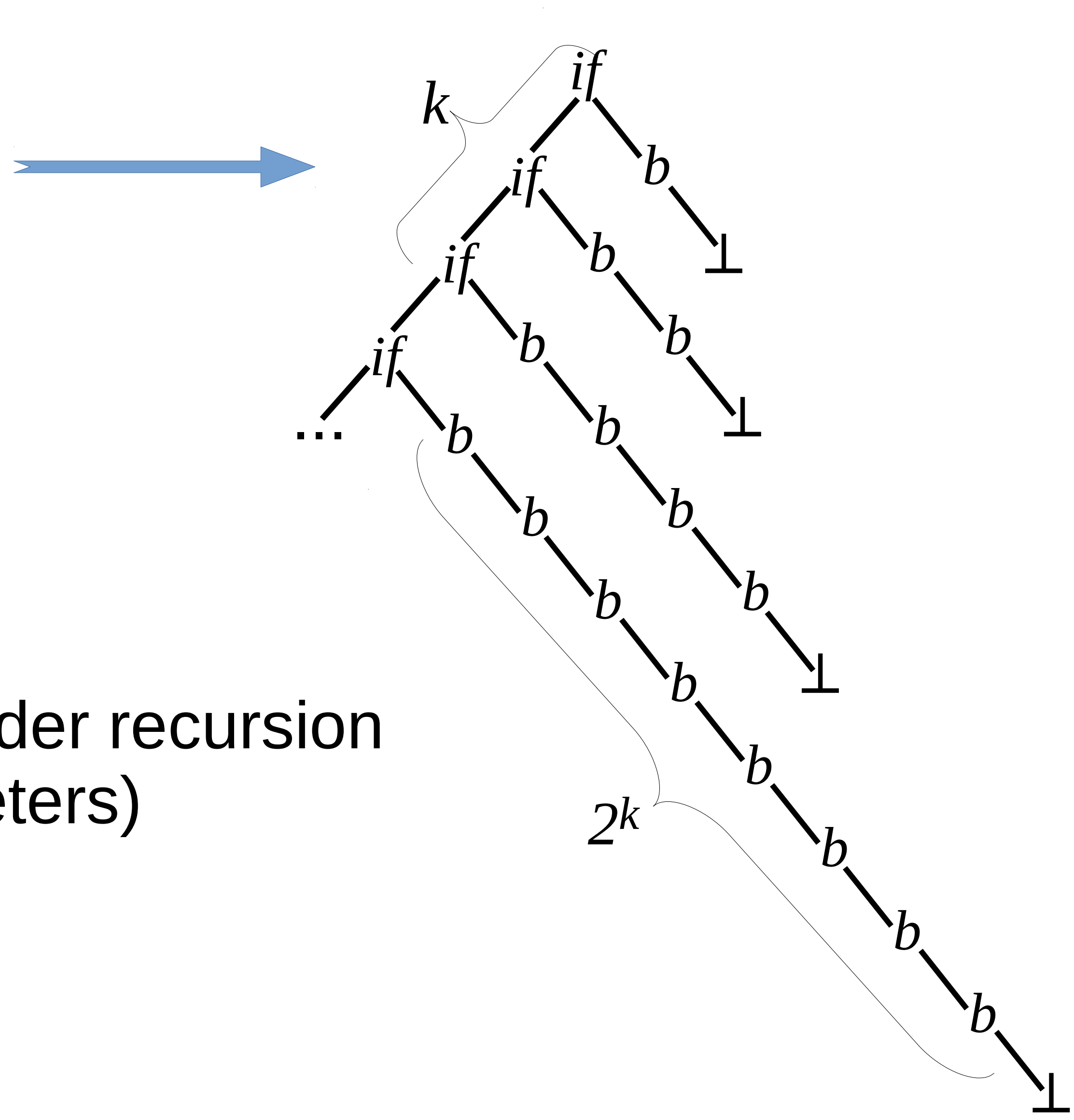


We are interested in trees representing the control flow of such programs.

Observation: these trees need not to be regular

Higher-order recursion schemes – example

```
fun A(f,x) {
   if * then A(D(f),x) else f(x);
}
fun D(f)(x) {
   f(x); f(x);
}
fun P(x) {
   b(x);
}
A(P,x)
```



This program uses higher-order recursion (passes functions as parameters)

Model-checking

Theorem [Ong 2006]
MSO model-checking on trees generated by recursion schemes is decidable.

Input: recursion scheme $\mathcal{G}$, MSO formula ϕ

Question: is ϕ true in the (infinite) tree generated by $\mathcal{G}$?

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This procedure can be used for model-checking programs written in functional programming languages:

Input: a program P, a property ψ
Question: does P satisfy ψ?

CEGAR loop, etc.
There exist tools that take (short) programs in Ocaml and can verify some useful properties.

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We consider the WMSO+U logic.

"+U" = we add a new quantifier „U" [Bojańczyk, 2004]

$$UX.\phi(X)$$

$\phi(X)$ holds for finite sets of arbitrarily large size

$$\forall n \in \mathbb{N}\ \exists X\ (\ n<|X|<\infty\ \wedge\ \phi(X)\ )$$

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“W” = weak – we can quantify only over finite sets
($\exists X$ / $\forall X$ means: exists a <u>finite</u> set X / for all <u>finite</u> sets X)

Decision problems for MSO+U

Satisfiability (the problem usually considered for MSO+U):
input: formula ϕ, question: is ϕ true in some tree?

- undecidable for MSO+U, even for words [Bojańczyk, P., Toruńczyk 2016]
  some fragments of MSO+U decidable for words [Bojańczyk, Colcombet 2006]
- decidable for WMSO+U [Bojańczyk, Toruńczyk 2012]
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HORS model-checking
input: formula ϕ, HORS $\mathcal{G}$,
question: is ϕ true in the tree generated by $\mathcal{G}$

- decidable for $\phi \in$MSO [Ong 2006]
- undecidable for $\phi \in$MSO+U (generalizes satifiability)
- **Contribution: decidable for $\phi \in$WMSO+U**

About the proof of model checking for WMSO+U

Theorem – the following problem is decidable:

input: formula $\phi \in$WMSO+U, HORS $\mathcal{G}$,

question: is ϕ true in the tree generated by $\mathcal{G}$?

Key ingredients:

- decidability of the (single-letter) “diagonal problem” for HORSes:

  input: HORS $\mathcal{G}$, letter a

  question: are there paths with arbitrarily many letters a in the tree generated by $\mathcal{G}$?

  [Hague, Kochems, Ong 2016, Clemente, P., Salvati, Walukiewicz 2016]

  Remark 1: this property is not regular

  Remark 2: this is a „universal” property that can be expressed by a single „U” quantifier

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- „reflection” for the diagonal problem: [P. 2016]
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- induction on the structure of ϕ – we add labels with information about subformulas (here it is useful that the logic is „weak")

Future work

The diagonal problem for HORS is decidable in a more general version:

input: HORS $\mathcal{G}$, letters $a_1,...,a_k$

question: are there paths with arbitrarily many appearances of every letter $a_1,...,a_k$ in the tree generated by $\mathcal{G}$?

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Question: Design a more general logic, capable to express the multi-letter diagonal problem (and prove its decidability for trees generated by HORSes, via a reduction to this version of the diagonal problem)

Thank you!