On computability and tractability for infinite sets

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The goal of this work is to identify the notion of computability, including "polynomial time computability", for definable sets. Definable sets are a generalisation of hereditarily finite sets. They are possibly infinite, but can be defined using set builder notation in terms of some underlying relational structure $\mathbb{A}$, called the *atoms* of the definable set. We begin with some examples. Suppose that the underlying structure of atoms is the natural numbers with equality $(\mathbb{N}, =)$. One possible definable set consists of all unordered pairs of atoms:

$$\{\{x, y\} : \text{ for } x, y \in \mathbb{A} \text{ such that } x \neq y\}.$$

We can use parameters from the atoms, e.g. as in the following definable set:

$$\{x : \text{ for } x \in \mathbb{A} \text{ such that } x \neq 5\}.$$

Another example is the set $\mathbb{A}^2$ of all ordered pairs, encoded via Kuratowski pairing:

$$\{\{x, \{x, y\}\} : \text{ for } x, y \in \mathbb{A} \text{ such that true}\}$$

If the atoms have more structure, then this structure can be used in the definable sets, e.g. if the atoms are the ordered rational numbers $(\mathbb{Q}, \leq)$ then an example of a definable set is the set of all closed intervals with right endpoint ≤ 7:

$$\{\{y : \text{ for } y \in \mathbb{A} \text{ such that } x \leq y \wedge y \leq z\} : \text{ for } x, y \in \mathbb{A} \text{ such that } x \leq y \wedge y \leq 7\}$$

Definable sets are a flexible and easy to use formalism for representing some possibly infinite data structures. The goal of this work is to define what it means for an operation on definable sets to be computable. A second goal, and the main original contribution, is to propose a definition of "polynomial time" computation. We propose such a notion, in the spirit of fixed parameter tractability, and call it *fixed dimension polynomial* computation.

Finally, we show that a programming language manipulating definable sets, called *definable while programs*, when restricted suitably, defines fixed dimension polynomial computations. The questions whether definable while programs, or any "logic" whatsoever capture all fixed dimension polynomial computations, are closely related to analogous questions from Finite Model Theory, namely, whether $\tilde{\text{C}}\text{PT}$, or any logic whatsoever captures polynomial time computation over finite structures.