Quantitative Reductions and Vertex-Ranked Games*

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We introduce quantitative reductions, a novel technique for solving quantitative games that does not rely on a reduction to qualitative games, and demonstrate that quantitative reductions exhibit the same desirable properties as qualitative ones. Additionally, they retain the optimality of solutions. We introduce vertex-ranked games as a general-purpose target for quantitative reductions. In such games the value of a play is determined only by a qualitative winning condition and a ranking of the vertices. We demonstrate how to solve such games optimally and we use this framework to solve quantitative request-response games and quantitative Muller games optimally.

The study of quantitative infinite games has garnered great interest lately, since they allow for a much more fine-grained analysis and specification of reactive systems than classical qualitative games. While such games have been investigated previously, the algorithms handling such games usually rely on ad-hoc solutions that are tailor-made to the problem under consideration. In particular, no general tools have been developed for the analysis of such games. Our framework, for the first time, disentangles the study of quantitative games from that of qualitative ones.

Qualitative infinite games have been applied successfully in the verification and synthesis of reactive systems and have given rise to a multitude of algorithms that ascertain system correctness and that synthesize correct-by-construction systems. In such a game, two players, called Player 0 and Player 1, respectively, move a token in a directed graph. After infinitely many moves, one player is declared as the winner of the resulting play. As an example, in a qualitative request-response game, Player 0 wants to ensure that every visit to a vertex denoting some request is eventually followed by a visit to a vertex denoting an answer to that request. To solve qualitative games, i.e., to determine a winning strategy for one player, one often reduces a complex game to a potentially larger, but conceptually simpler one. For example, in a multi-dimensional request-response game, i.e., a request-response game in which there exist multiple conditions that can be requested and answered, one stores the set of open requests and demands that every request is closed at infinitely many positions. As this is a Büchi condition, which is much simpler than the request-response condition, one is able to reduce request-response games to Büchi games of exponential size.

In recent years, the focus of research has shifted from the study of qualitative games, in which one player is declared as the winner, to that of quantitative games, in which the resulting play is associated with some value. Such games allow to model systems in which, e.g., requests have to be answered within a certain number of steps [2, 5, 7, 9], systems with one or more finite resources which may be drained and charged [1, 3, 4, 10], or scenarios in which each move incurs a certain cost for either player [6, 8, **?**].

In general, one player aims to minimize the value of the resulting play, while the other one seeks to maximize it. In a quantitative request-response game, for example, it is the goal of Player 0 to minimize the number of steps between requests and their corresponding answers. The typical questions asked in the context of such games are "Can Player 0 bound the maximal time between requests and responses?" [5, 8, 9], "What is the minimal time between requests and responses that Player 0 can ensure?" [12], "What is the minimal average level of the resource that Player 0 can ensure without it ever running out?" [1], or "Can Player 0 ensure an average cost per step greater than zero?" [13].

Such decision problems are usually answered by reducing the quantitative game to a qualitative threshold game, where some bound is hardcoded during the reduction. If the value of the resulting play satisfies the bound, then Player 0 is declared as the winner. For example, in order to determine the winner in a quantitative request-response game as described above for a given bound b, we construct a Büchi game in which every time a request is opened, a counter is started which counts up to the bound b and is reset if the request is answered. Once a counter exceeds the value b, Player 0 loses the play. We then require that every counter is not running infinitely often, which is again a Büchi condition, i.e., it is much simpler than the original quantitative request-response condition. The resulting game is won by Player 0 if and only if she can ensure that every request is answered within at most b steps.

These reductions are usually very specific to the problem being addressed. Furthermore, they abandon the quantitative aspect of the game under consideration, as the bound is hardcoded during the reduction.

* Extended abstract of [11]

Thus, even when only changing the bound one is interested in, the reduction has to be recomputed and the resulting game has to be solved from scratch. In the request-response example, if one is interested in checking whether Player 0 can ensure every request to be answered within at most $b' \neq b$ steps, one would construct a new Büchi game. This game would then be solved independently of the previously computed one for the bound b.

In this work, we lift the concept of reductions from qualitative games to quantitative games. Quantitative reductions enable the study of a multitude of optimization problems for quantitative games similarly to decision problems for qualitative games. When investigating quantitative request-response games using quantitative reductions, for example, we only compute a single, simpler quantitative game. We can then easily check this game for a winning strategy for Player 0 for any bound b. If she has such a strategy in the latter game, we obtain a strategy satisfying the same bound in the former one via the reduction.

Thus, we retain the intuitive property of reductions for qualitative games: The properties of a complex quantitative game can be studied by investigating a potentially larger, but conceptually simpler quantitative game.

In order to obtain a backend for quantitative reductions, we define vertex-ranked games, a very general class of quantitative games, which can be used as targets for quantitative reductions. Such games are very simple quantitative games, in which the cost of a play is given only by a qualitative winning condition and by a ranking of the vertices of the game with natural numbers. If the resulting play is winning according to the qualitative condition, then its value is given by the highest rank visited at all or by the highest one visited infinitely often, depending on the specific variant of vertex-ranked games considered. Otherwise, the value of the play is infinite.

Finally, we demonstrate quantitative reductions from quantitative request-response games and quantitative Muller games to vertex-ranked games. These reduction, together with our algorithms for solving vertex-ranked games optimally, provide a simplified proof of the ExpTime-membership of the problem of solving quantitative Muller games. Moreover, we obtain ExpTime-completeness of the problem of solving quantitative request-response games optimally.

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