# A Crevice on the Crane Beach: Finite-Degree Predicates

**Michaël Cadilhac** **University of Oxford** michael@cadilhac.name
***Joint work with Charles Paperman, LICS'17 [2]***

## 1 Context

This study takes place in the context of first-order logic on words. Therein, a formula, say:

$$(\exists x)[x \equiv_2 0 \wedge \mathbf{a}(x)]$$

is true of a word $w$ if it holds on the structure with universe $\{1, \ldots, |w|\}$, with $\mathbf{a}$ interpreted as the set of positions $i$ for which $w_i = a$, and with pre-determined interpretations for the *numerical predicates* (here, $\equiv_2$). The above formula expresses that there is a letter $a$ at an even position in the input word.

## 2 Finite-degree predicates are almost all predicates

In a first, quite easy step, we show that all languages of full first-order logic (that is, when the numerical predicates are unrestricted) are expressible in first-order logic where the only numerical predicates at hand are:

- The linear order;
- The predicate $\mathrm{MSB}_0$, true of $(x, y)$ if $y$ is $x$ with its most-significant bit zeroed;
- The *finite-degree* predicates, i.e., the predicates that are subsets of $\mathbb{N}^k$ for which each integer appears a finite number of time in a tuple.

  In symbol, we write:

  $$\mathrm{FO}[\mathcal{A}_{\mathcal{RB}}] = \mathrm{FO}[\leq, \mathrm{MSB}_0, \mathcal{F}_{\mathcal{IN}}] \ .$$

## 3 Finite-degree predicates are fooled by neutral letters

For the second step, say that a language admits a neutral letter if there is a letter that can be added or removed from any word without impacting membership. The Crane Beach Property, introduced more than a decade ago [1], is true of a logic if all the expressible languages admitting a neutral letter are regular. This corresponds to the intuition that all the numerical predicates are rendered useless by the presence of a neutral letter.

Although it is known that $\mathrm{FO}[\mathcal{A}_{\mathcal{RB}}]$ does not have the Crane Beach Property [1], we will see that that the (strong form of the) Crane Beach Property holds for both $\mathrm{FO}[\leq, \mathcal{F}_{\mathcal{IN}}]$ and $\mathrm{FO}[\leq, \mathrm{MSB}_0]$. Thus $\mathrm{FO}[\leq, \mathcal{F}_{\mathcal{IN}}]$ exhibits a form of locality and the Crane Beach Property, and can still express a wide variety of languages, while being one simple predicate away from the expressive power of $\mathrm{FO}[\mathcal{A}_{\mathcal{RB}}]$.

## 4 On counting

As an application, we will study the counting abilities of $\mathrm{FO}[\leq, \mathcal{F}_{\mathcal{IN}}]$. We say that a logic can count up to $f(n)$ if there is a formula that is true of $k < f(n)$ on words of size $n$ iff the word has exactly $k$ letters $a$. It is known that counting to $\log(\log(\cdots(\log(n))))$ implies that the Crane Beach Property does not hold. We show the even stronger statement that $\mathrm{FO}[\leq, \mathcal{F}_{\mathcal{IN}}]$ cannot count more than a constant.

## References

1 David A. Mix Barrington, Neil Immerman, Clemens Lautemann, Nicole Schweikardt, and Denis Thérien. First-order expressibility of languages with neutral letters or: The Crane Beach Conjecture. *J. Computer and System Sciences*, 70:101–127, 2005.

2 Michaël Cadilhac and Charles Paperman. A crevice on the Crane Beach: Finite-degree predicates. In *Logic in Computer Science - 32nd Annual ACM/IEEE Symposium, LICS 2017, Reykjavik, Iceland (to appear)*, 2017.