PERFECT HALF SPACE GAMES

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A d-dimensional energy game [4, 11] sees two players compete in a finite game graph, whose edges are decorated with vectors of weights in $\mathbb{Z}^d$. The d weights represent various discrete resources that can be consumed or replenished by the actions of the game. The objective of Player 1, given an initial credit in $\mathbb{N}^d$, is to play indefinitely without depleting any of the resources—more precisely to keep the current sum of encountered weights plus initial credit non-negative in every dimension—while Player 2 attempts to foil this. The primary motivation for these games is controller synthesis for resource-sensitive reactive systems, where they are also closely related to multi-dimensional mean-payoff games—and actually equivalent if finite-memory strategies are sought for the latter [11, Lemma 6]. But they appear in diverse settings: for example, in process algebra, they are equivalent to the simulation problem between a finite state system and a Petri net or a basic parallel process [8, Propositions 6.2 and 6.4]; in artificial intelligence, they allow to solve the model-checking problem for the resource-bounded logic RB±ATL [3, 2].

The algorithmic issues surrounding multi-dimensional energy games have come under considerable scrutiny. Deciding whether there exists an initial credit that would allow Player 1 to win is coNP-complete [11, Theorem 3], while the complexity when the initial credit is given as part of the input becomes 2-EXPTIME-complete [8, 10]. Finally, both decision problems are in pseudo-polynomial time when d is fixed [10].

Open Questions. However, these recent advances do not settle the case of multi-dimensional energy parity games [5], where Player 1 must ensure that, in addition to the quantitative energy objective (specifying resource consumption and replenishment), she also complies with a qualitative ω-regular objective in the form of a parity condition (specifying functional requirements). These games with arbitrary initial credit are still coNP-complete as a consequence of [5, Lemma 4]. With given initial credit, they were first proven decidable by Abdulla, Mayr, Sangnier, and Sproston [1], and used to decide both the model-checking problem for a suitable fragment of the μ-calculus against Petri net executions and the *weak* simulation problem between a finite state system and a Petri net; they also allow to decide the model-checking problem for the resource logic RB±ATL* [2]. As shown by Jančar [9], d-dimensional energy games using $2p$ priorities can be reduced

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to 'extended' multi-dimensional energy games of dimension $d' \stackrel{\text{def}}{=} d + p$, with complexity upper bounds shown earlier by Brázdil, Jančar, and Kučera [4] to be in $(d'-1)$-EXPTIME when $d' \geq 2$ is fixed, and in TOWER when d' is part of the input, leaving a substantial complexity gap with the 2-EXPTIME-hardness shown in [8].

Contributions. We introduce *perfect half space games*, both

- as intermediate objects in a chain of reductions from multi-dimensional energy parity games to mean-payoff games, allowing us to derive new tight complexity upper bounds based on recent advances by Comin and Rizzi [7] on the complexity of mean-payoff games, and
- as a means to gain a deeper understanding of how winning strategies in energy games are structured.

More precisely, in perfect half-space games, positions are pairs: a vertex from a d-dimensional game graph as above, together with a d-dimensional *perfect half space*. The latter is a maximal salient blunt cone in $\mathbb{Q}^d$: a union of open half spaces of dimensions d, $d-1$, ..., 1, where each is contained in the boundary of the previous one. In these games, Player 1 may not change the current perfect half space, but Player 2 may change it arbitrarily at any move. However, the goal of Player 2 is to make the sums of encountered weights diverge in a direction which is consistent with the chosen perfect half spaces; thus the greater the dimension of the component open half spaces that Player 2 varies infinitely often, the harder it is for him to win. For example, with $d = 2$, if Player 2 eventually settles on the perfect half space that consists of the half plane $x < 0$ and the half line $x = 0 \wedge y < 0$, then he wins provided the sequence of total weights is such that either their x-coordinates diverge to $-\infty$, or their x-coordinates do not diverge to $+\infty$ and their y-coordinates diverge to $-\infty$; if however Player 2 switches between the two half lines of $x = 0$ infinitely often, then he can only win in the former manner.

Firstly, we show that perfect half space games can be easily translated to the *lexicographic energy games* of Colcombet and Niwiński [6]. The translation amounts to normalising the edge weights with respect to the current perfect half spaces, and inserting another d dimensions in which we encode appropriate penalties for Player 2 that are imposed whenever he changes the perfect half space. We deduce that perfect half space games are positionally determined, and moreover that Player 2 has winning strategies that are oblivious to the current perfect half space. Along the way, we provide a proof of the positional determinacy of lexicographic energy games, along with pseudo-polynomial complexity upper bounds for their decision problem when d is fixed, based on the recent results of Comin and Rizzi [7] for mean-payoff games.

Secondly, we establish that perfect half space games capture *bounding games*. The latter were central to obtaining the tight complexity upper bounds for multi-dimensional energy games [10]. They are played purely on the d-dimensional game graphs and have a simple winning condition: the goal of Player 1 is to keep the total absolute value of weights bounded (i.e., contained in some d-dimensional hypercube). One reading of this reduction is that whenever Player 2 has a winning strategy in a bounding game, he has one that 'announces' at every move some perfect half space and succeeds in forcing the total weights to be unbounded in a direction consistent with the infinite sequence of his announcements. The proof is difficult, and relies on a construction from the previous paper [10] of a winning strategy for Player 1 in the bounding game given her winning strategy in a first-cycle game featuring perfect half spaces. Composing this with our complexity bounds for lexicographic energy games gives us a new approach to solving bounding games, improving the time complexity from the previously best $(|V| \cdot \|E\|)^{O(d^4)}$ [10, Corollary 3.2] to $(|V| \cdot \|E\|)^{O(d^3)}$, where V is the set of vertices and $\|E\|$ the maximal absolute value over the weights in the input multi-dimensional game graph.

Thirdly, building on Jančar's reduction, we show how multi-dimensional energy parity games can be solved by means of bounding games. For the given initial credit problem, we obtain 2-EXPTIME-completeness, closing the aforementioned complexity gap. When the dimension d and the number of priorities $2p$ are fixed, we obtain that, for both arbitrary and given initial credits, the winner is decidable in pseudo-polynomial time. With arbitrary initial credit, our new bound $(|V| \cdot \|E\|)^{O((d+p)^3 \log(d+p))}$ improves when $p = 0$ over the previously best $(|V| \cdot \|E\|)^{O(d^4)}$ [10, Theorem 3.3].

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