Games with lexicographically ordered ω-regular objectives

Véronique Bruyère[1], Quentin Hautem[1], Antonia Lechner[2], and Jean-François Raskin[2]

[1] Département d'informatique, Université de Mons (UMONS), Belgium
[2] Département d'informatique, Université libre de Bruxelles (ULB), Belgium

Abstract. In recent years, two-player zero-sum games with multidimensional objectives have received a lot of interest as a model for intricate systems that are required to satisfy several properties. In this framework, player 1 wins if he can ensure that all objectives are satisfied against any behavior of player 2. It is however often natural to provide more significance to one objective over another, a situation that can be modeled by lexicographically ordering the objectives. Inspired by recent work on concurrent games with ω-regular objectives by Bouyer et al., we investigate in detail turned-based lexicographic games with all the classically studied ω-objectives. Given a tuple of objectives, the payoff associated with a given play of the game is a Boolean vector, the components of which indicate which objectives are satisfied. We study the threshold problem which asks whether player 1 can ensure a payoff greater than or equal to a given Boolean threshold vector w.r.t. the lexicographic order. We provide precise results that refine and complete the ones by Bouyer et al. for turn-based games, including exact complexity classes, deterministic algorithms for computing the values and the optimal strategies, and memory requirements for those strategies. Whereas the threshold problem is PSPACE-complete for several ω-regular objectives, we show that it is fixed parameter tractable in those cases.

1 Introduction

Two-player zero-sum games played on directed graphs constitute an important framework for the synthesis of a suitable controller for a reactive system facing an uncontrollable environment [7]. To model properties to be ensured in the reactive system against the environment, games with Boolean objectives and games with quantitative objectives have been studied, for example games with ω-regular objectives [6] and mean-payoff games [9].

In recent years, games with multidimensional objectives have received a lot of interest as they are more adapted to model intricate systems with several properties to be satisfied. In this framework, player 1 wins if he can ensure that all objectives are satisfied against any behavior of player 2. Generalized parity games [5] and multi-mean-payoff games studied in [8] with the same kind of objective on each dimension are examples of such games. Very recently, multi-dimensional games with heterogeneous objectives have also been investigated, see [3].

Another natural way of modeling a system that must ensure several properties is to provide more significance to one objective over another. Imagine the situation where one wants a system with high performance and low energy consumption which is required to stay below a certain temperature and to stay operational. The "non-overheating" objective clearly has priority over the other objectives of high performance and low energy consumption. A way to model such a classification of the objectives is to order them lexicographically. Lexicographic games with mean-payoff objectives are studied in [1]. Among several orders, the lexicographic order is also investigated in [2] for concurrent games with ω-regular objectives.

In this ongoing work, we consider turned-based lexicographic games with the classical ω-regular objectives: reachability, safety, Büchi, co-Büchi, parity, Streett, Rabin, explicit Muller, and Muller objectives. Here, a lexicographic game is a game with multidimensional objectives of the same kind, and where with every play we associate a Boolean vector whose component is 1 if the play satisfies the corresponding objective, and 0 otherwise. In such games, player 1 aims at maximizing his payoff with respect to the lexicographic order, while player 2 aims at minimizing it. We study the threshold problem, which asks, given some threshold, whether player 1 has a winning strategy to ensure a payoff greater than or equal to the threshold with respect to the lexicographic order.

Our contributions are as follows (see Table 1). First, we give a full picture of the complexity of the threshold problem for each kind of ω-regular objective. Complexity upper bounds follow from results of [2]. We provide additional tight lower bounds and deterministic algorithms for the threshold problem. We also study optimal strategies, that is, strategies that guarantee the highest (resp. lowest) possible threshold, called value, for player 1 (resp. player 2). We describe how to compute values and optimal strategies, and provide tight memory bounds for those strategies. Very recently, Calude et al. provided a quasipolynomial-time algorithm for parity games and showed that parity games are fixed parameter tractable [4]. Using this result, we show that the threshold problem is also fixed parameter tractable for all considered ω-regular objectives.

<table>
<tr><th></th><th>Reach</th><th>Safe</th><th>Buchi</th><th>CoBuchi</th><th>ExplMuller</th><th>Parity</th><th>Streett</th><th>Rabin</th><th>Muller</th></tr>
<tr><td>Threshold problem</td><td colspan="2">PSPACE-complete</td><td colspan="3">P-complete</td><td colspan="4">PSPACE-complete</td></tr>
<tr><td>Optimal strategies</td><td colspan="2" rowspan="2">exponential</td><td colspan="2" rowspan="2">polynomial</td><td>exponential</td><td colspan="4" rowspan="2">exponential</td></tr>
<tr><td>Value</td><td>polynomial</td></tr>
<tr><td>Memory of $\mathcal{P}_1$'s optimal strategies</td><td colspan="2" rowspan="2">exponential</td><td>linear</td><td>memoryless</td><td colspan="5" rowspan="2">exponential</td></tr>
<tr><td>Memory of $\mathcal{P}_2$'s optimal strategies</td><td>memoryless</td><td>linear</td></tr>
</table>

Table 1. Overview of the results on lexicographic games

References

1. R. Bloem, K. Chatterjee, T. A. Henzinger, and B. Jobstmann. Better quality in synthesis through quantitative objectives. In A. Bouajjani and O. Maler, editors, *Computer Aided Verification, 21st International Conference, CAV 2009, Grenoble, France, June 26 - July 2, 2009. Proceedings*, volume 5643 of *Lecture Notes in Computer Science*, pages 140–156. Springer, 2009.
2. P. Bouyer, R. Brenguier, N. Markey, and M. Ummels. Concurrent games with ordered objectives. In L. Birkedal, editor, *Foundations of Software Science and Computational Structures - 15th International Conference, FOSSACS 2012, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2012, Tallinn, Estonia, March 24 - April 1, 2012. Proceedings*, volume 7213 of *Lecture Notes in Computer Science*, pages 301–315. Springer, 2012.
3. V. Bruyère, Q. Hautem, and J. Raskin. On the complexity of heterogeneous multidimensional games. In J. Desharnais and R. Jagadeesan, editors, *27th International Conference on Concurrency Theory, CONCUR 2016, August 23-26, 2016, Québec City, Canada*, volume 59 of *LIPIcs*, pages 11:1–11:15. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2016.
4. C. Calude, S. Jain, B. Khoussainov, W. Li, and F. Stephan. Deciding parity games in quasi polynomial time. *STOC, 2017. To appear*, 2017.
5. K. Chatterjee, T. A. Henzinger, and N. Piterman. Generalized parity games. In H. Seidl, editor, *Foundations of Software Science and Computational Structures, 10th International Conference, FOSSACS 2007, Held as Part of the Joint European Conferences on Theory and Practice of Software, ETAPS 2007, Braga, Portugal, March 24-April 1, 2007, Proceedings*, volume 4423 of *Lecture Notes in Computer Science*, pages 153–167. Springer, 2007.
6. E. Grädel, W. Thomas, and T. Wilke, editors. *Automata, Logics, and Infinite Games: A Guide to Current Research*, volume 2500 of *Lecture Notes in Comput. Sci.* Springer, 2002.
7. A. Pnueli and R. Rosner. On the synthesis of a reactive module. In *POPL*, pages 179–190. ACM Press, 1989.
8. Y. Velner, K. Chatterjee, L. Doyen, T. A. Henzinger, A. M. Rabinovich, and J. Raskin. The complexity of multi-mean-payoff and multi-energy games. *Inf. Comput.*, 241:177–196, 2015.
9. U. Zwick and M. Paterson. The complexity of mean payoff games on graphs. *Theor. Comput. Sci.*, 158:343–359, 1996.