On Strong Determinacy of Countable Stochastic Games *

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Abstract

We study 2-player turn-based perfect-information stochastic games with countably infinite state space. The players aim at maximizing/minimizing the probability of a given event (i.e., measurable set of infinite plays), such as reachability, Büchi, ω-regular or more general objectives.

These games are known to be weakly determined, i.e., they have value. However, strong determinacy of threshold objectives (given by an event $\mathcal{E}$ and a threshold $c \in [0,1]$) was open in many cases: is it always the case that the maximizer or the minimizer has a winning strategy, i.e., one that enforces, against all strategies of the other player, that $\mathcal{E}$ is satisfied with probability $\geq c$ (resp. $< c$)?

We show that almost-sure objectives (where $c = 1$) are strongly determined. This vastly generalizes a previous result on *finite* games with almost-sure tail objectives. On the other hand we show that $\geq 1/2$ Büchi and co-Büchi objectives are not strongly determined, not even if the game is finitely branching.

Moreover, for almost-sure reachability and almost-sure Büchi objectives in finitely branching games, one of the players must have a memoryless deterministic (MD) winning strategy.

Weak determinacy. Using Martin's result [4], Maitra & Sudderth [3] showed that stochastic games with Borel payoffs are *weakly determined*, i.e., all states have value.

Strong determinacy. Given a predicate $\mathcal{E}$ on plays and a constant $c \in [0,1]$, strong determinacy of a threshold objective $(\mathcal{E}, \triangleright c)$ (where $\triangleright \in \{>, \geq\}$) holds iff either the maximizer or the minimizer has a winning strategy, i.e., a strategy that enforces (against any strategy of the other player) that the predicate $\mathcal{E}$ holds with probability $\triangleright c$ (resp. $\not\triangleright\ c$). In the case of $(\mathcal{E}, = 1)$, one speaks of an almost-sure $\mathcal{E}$ objective. If the winning strategy of the winning player can be chosen MD (memoryless deterministic) then one says that the threshold objective is strongly MD determined. Similarly for other types of strategies, e.g., FR (finite-memory randomized).

Our contribution to determinacy. We show that almost-sure Borel objectives are strongly determined for games with *countably infinite* state spaces. (In particular this even holds for infinitely branching games; cf. Table 1.) This removes both the restriction to finite games and the restriction to tail objectives of [1, Theorem 3.3], and solves an open problem stated there.

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Objective	> 0	$> c$	$\geq c$	$= 1$
Reachability	✓(MD)	✓(MD)	✓(¬FR)	✓**(MD)**
Büchi	✔(¬FR)	✖	✖	✔**(MD)**
Borel	✔(¬FR)	✖	✖	✔(¬FR)

(a) Finitely branching games

Objective	> 0	$> c$	$\geq c$	$= 1$
Reachability	✓(MD)	×	×	✔(¬FR)
Büchi	✔(¬FR)	×	×	✔(¬FR)
Borel	✔(¬FR)	×	×	✔(¬FR)

(b) Infinitely branching games

Table 1: Summary of determinacy and memory requirement properties for reachability, Büchi and Borel objectives and various probability thresholds. The results for safety and co-Büchi are implicit, e.g., > 0 Büchi is dual to to $= 1$ co-Büchi. Similarly, (Objective, $> c$) is dual to ($\neg$Objective, $\geq c$). The results hold for every constant $c \in (0, 1)$. "✓(MD)" stands for "strongly MD-determined", "✓(¬FR)" stands for "strongly determined but not strongly FR-determined" and × stands for "not strongly determined". New results are in boldface.

On the other hand, we show that, for countable games, $\triangleright c$ (co-)Büchi objectives are not strongly determined for any $c \in (0, 1)$, not even if the game graph is finitely branching.

Our contribution to strategy complexity. While $\triangleright c$ reachability objectives in finitely branching countable games are not strongly MD determined in general [2], we show that strong MD determinacy holds for many interesting subclasses. In finitely branching games, it holds for strict inequality $> c$ reachability, almost-sure reachability, and in all games where either player $\Box$ does not have any value-decreasing transitions or player $\Diamond$ does not have any value-increasing transitions.

Moreover, we show that almost-sure Büchi objectives (but not almost-sure co-Büchi objectives) are strongly MD determined, provided that the game is finitely branching.

References

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