Decidable Weighted Expressions with Presburger Combinators*

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Abstract In this paper, we investigate the expressive power and the algorithmic properties of weighted expressions, which define functions from finite words to integers. First, we consider a slight extension of an expression formalism, introduced by Chatterjee. et. al. in the context of infinite words, by which to combine values given by unambiguous (max,+)-automata, using Presburger arithmetic. We show that important decision problems such as emptiness, universality and comparison are PSPACE-C for these expressions. We then investigate the extension of these expressions with Kleene star. This allows to iterate an expression over smaller fragments of the input word, and to combine the results by taking their iterated sum. The decision problems turn out to be undecidable, but we introduce the decidable and still expressive class of synchronised expressions.

Quantitative languages Quantitative languages (QL), or series, generalize Boolean languages to function from finite words into some semiring. They have recently received a particular attention from the verification community, for their application in modeling system *quality* [3], lifting classical Boolean verification problems to a quantitative setting. In this paper, we consider the case of integer weights and in this context, the comparison problem asks whether two QL $f, g : \Sigma^* \to \mathbb{Z}$ satisfy $f(u) \leq g(u)$ for all $u \in \Sigma^*$. Similarly, the universality ($f \geq \nu$ where ν is a constant) and equivalence problem ($f = g$) can be defined, as well as emptiness (does there exists a word whose value is above some given threshold). We say that a formalism for QL is decidable if all these problems are decidable. A popular formalism to define QL is that of weighted automata (WA) [5]. However, WA over the semiring $(\mathbb{Z}, max, +)$, called $(max, +)$-automata, are undecidable [9], even if they are linearly ambiguous $(max, +)$-automata [4].

Decidable formalisms for quantitative languages and objectives The largest known class of $(max, +)$-automata enjoying decidability is that of finitely ambiguous $(max, +)$-automata, which is also expressively equivalent to the class of finite-valued $(max, +)$-automata (all the accepting executions over the same input run yields a constant number of different values) [6]. Moreover, $(max, +)$-automata are not closed under simple operations such as min and the difference - [8]. Basic functions such as $u \mapsto \min(\#_a(u), \#_b(u))$ and[1] (as a consequence) $u \mapsto |f(u) - g(u)|$ are not definable by $(max, +)$-automata, even if f, g are [8].To cope with the expressivity and undecidability issues, a class of weighted expressions was introduced in [2] in the context of ω-words. Casted to finite words, the idea is to use (input) deterministic (max,+)-automata as atoms, and to combine them using the operations max, min, +, and -. The decision problems defined before were shown to be PSPACE-C [10] over ω-words. One limitation of this formalism, casted to finite words, if that it is not expressive enough to capture finitely ambiguous $(max, +)$-automata, yielding two incomparable classes of QL. In this paper, our objective is to push the expressiveness of weighted expressions as far as possible while retaining decidability, and to capture both finitely ambiguous (max,+)-automata and the expressions of [2], for finite words.

Monolithic expressions with Presburger combinators We define a class of expressions, inspired from [2], that we call monolithic in contrast to another class of expressions defined in a second contribution. The idea is to use unambiguous (max,+)-automata as atoms, and to com-

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[1] $\#_\sigma(u)$ is the number of occurrences of σ in u

bine them using n-ary functions definable in Presburger arithmetics (we call them Presburger combinators). Any finitely ambiguous (max,+)-automaton being equivalent to a finite union of unambiguous ones [6], this formalism captures finitely ambiguous (max,+)-automata (using the Presburger combinator max). We show that all the decision problems are PSPACE-C, matching the complexity of [10]. It is important to mention that this complexity result cannot be directly obtained from [10] which is on ω-words with mean-payoff automata as atoms (hence the value of an infinite word is prefix-independent). Moreover, unlike in [10], we can rely on existing results by encoding expressions into reversal-bounded counter machines [7].

Expressions with iterated sum The previous expressions are monolithic in the sense that first, some values are computed by weighted automata applied on the whole input word, and then these values are combined using Presburger combinators. It is not possible to iterate expressions on factors of the input word, and to aggregate all the values computed on these factors, for instance by a sum operation. The basic operator for iteration is that of Kleene star (extended to quantitative languages), which we call more explicitly *iterated sum*. It has already be defined in [5], and its unambiguous version considered in [1] to obtain an expression formalism equivalent to unambiguous (max,+)-automata. Inspired by [1], we investigate the extension of monolithic expressions with unambiguous iterated sum, which we just call iterated sum in the paper. The idea is as follows: given an expression E which applies on a domain D, the expression $E^{\#}$ is defined only on words u that can be uniquely decomposed (hence the name unambiguous) into factors $u_1 u_2 \dots u_n = u$ such that $u_i \in D$, and the value of u is then $\sum_{i=1}^{n} E(u)$. Unfortunately, we show that such an extension yields undecidability (if 2 or more iterated sum operations occur in the expression). The undecidability is caused by the fact that subexpressions $E^{\#}$ may decompose the input word in different ways. We therefore define the class of so called *synchronised* expressions with iterated sum, which forbids this behaviour. We show that while being expressive (for instance, they can define QL beyond finitely ambiguous (max,+)-automata), decidability is recovered. The proof goes via a new weighted automata model, called *weighted chop automata*, that slice the input word into smaller factors, recursively apply smaller chop automata on the factors to compute their values, which are then aggregated by taking their sum. In their synchronised version, we show decidability for chop automata.

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