Definability equals recognizability for graphs of bounded treewidth

Mikołaj Bojańczyk, Michał Pilipczuk

Highlights in Brussels,
September 8^{th}, 2016

Joint work with...

Mikołaj Bojańczyk, here receiving the **Crystal Brussels Sprout Award**.

How to construct graphs

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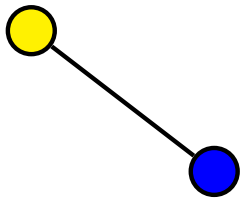

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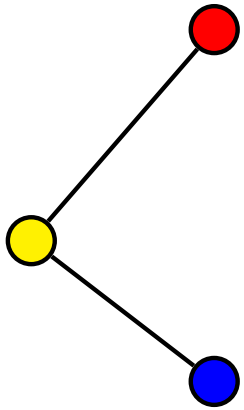

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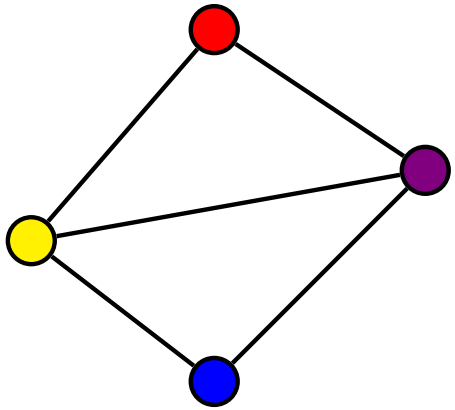

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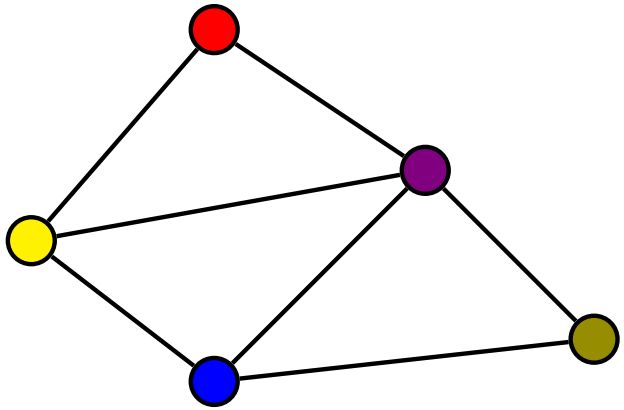

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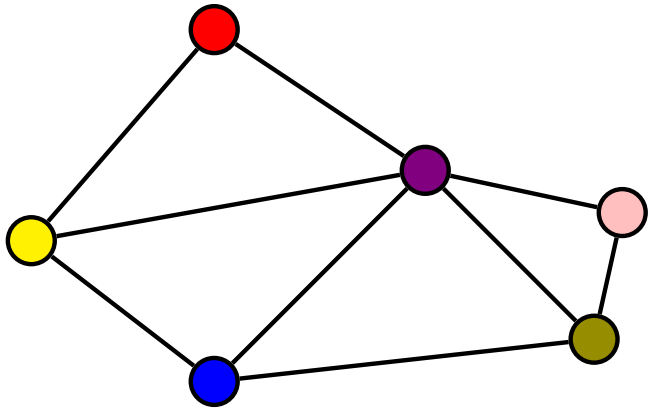

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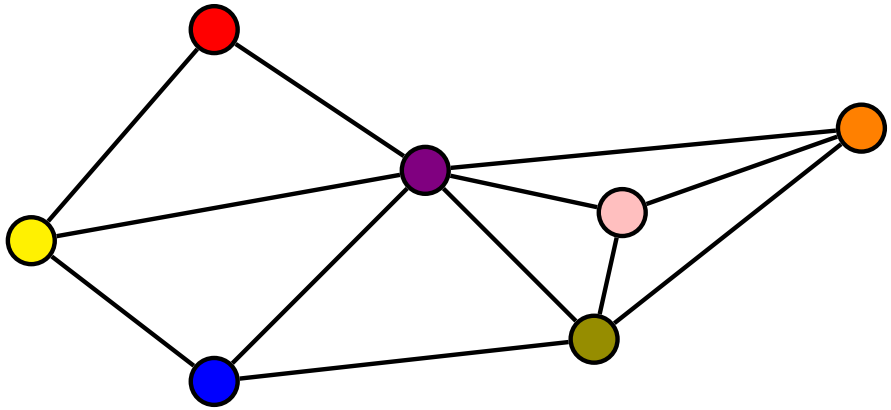

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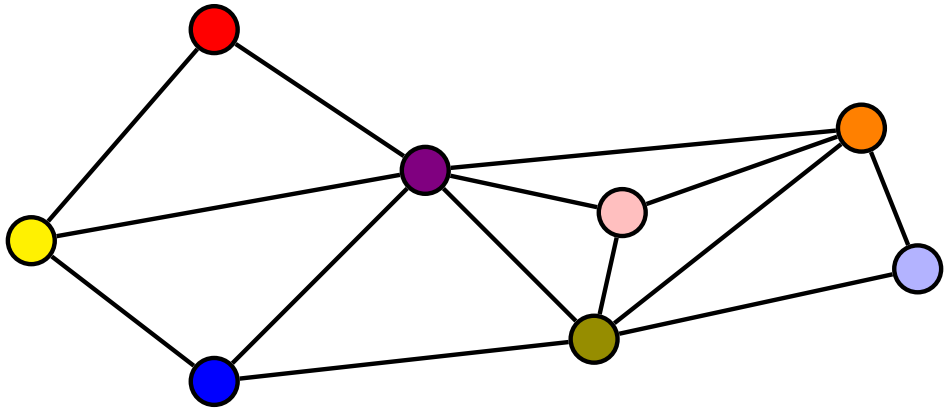

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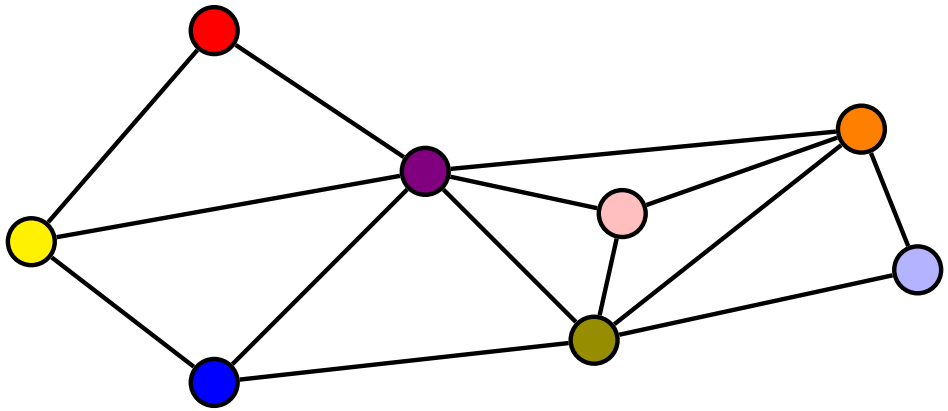

Idea: Keep only a small number of vertices in memory.

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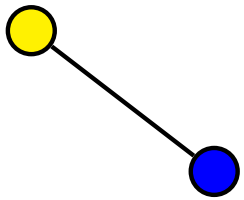

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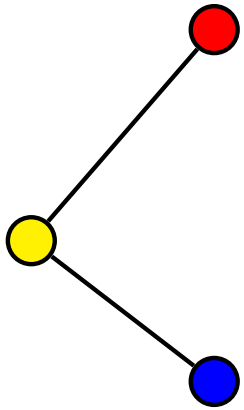

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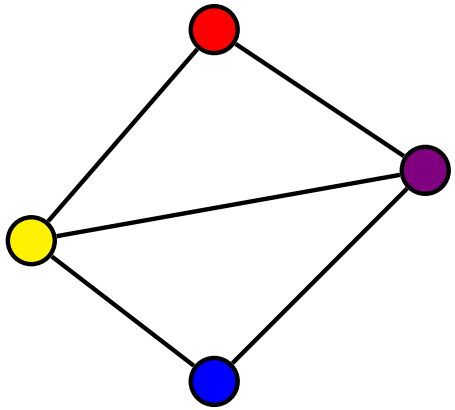

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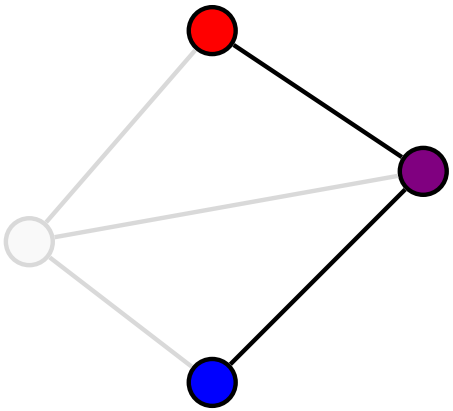

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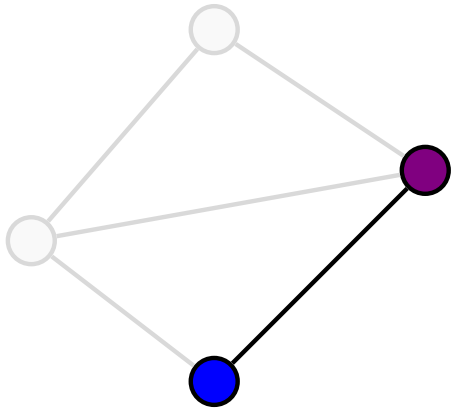

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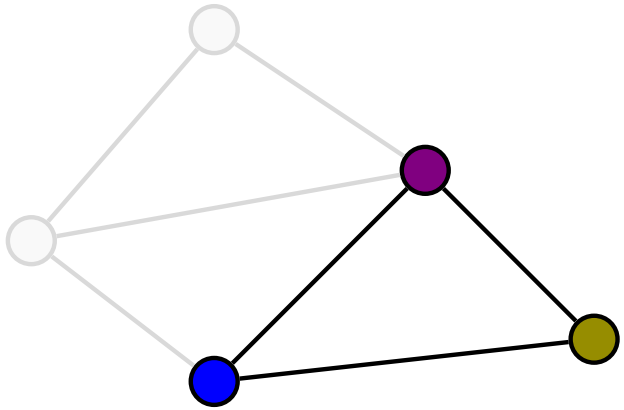

Introduce a new active vertex

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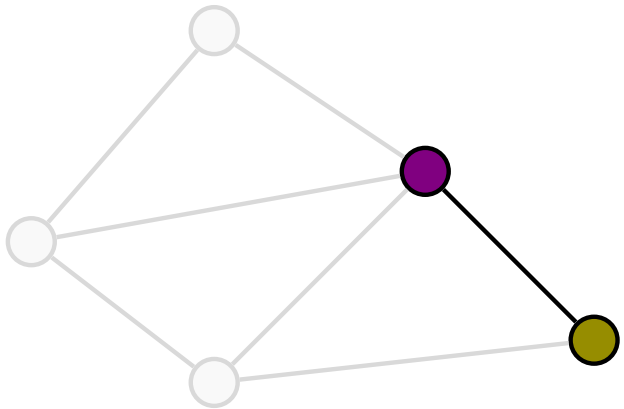

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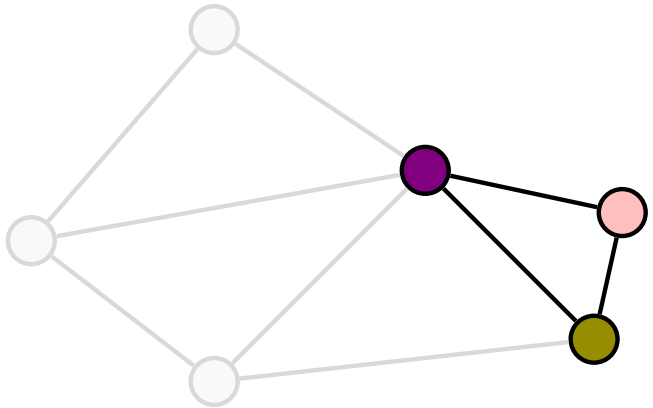

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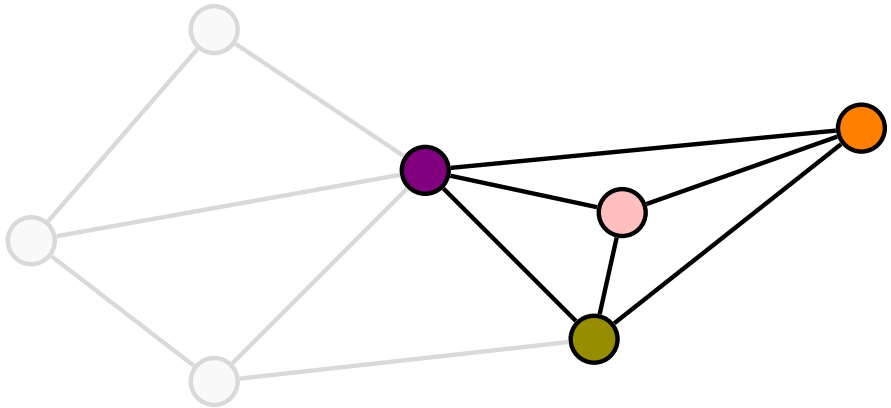

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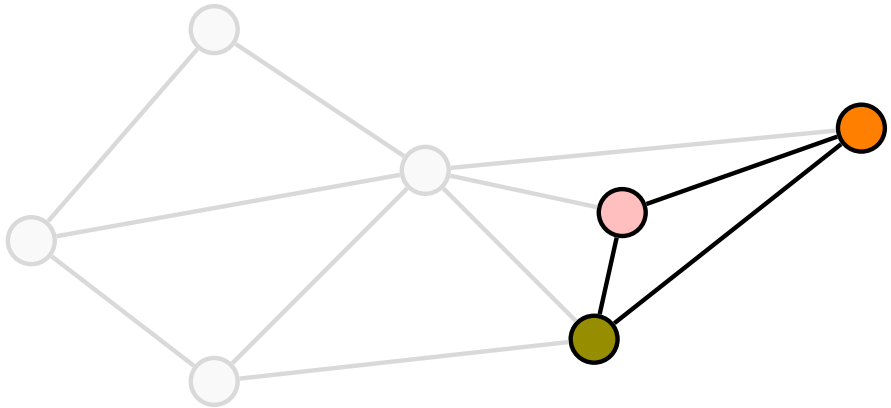

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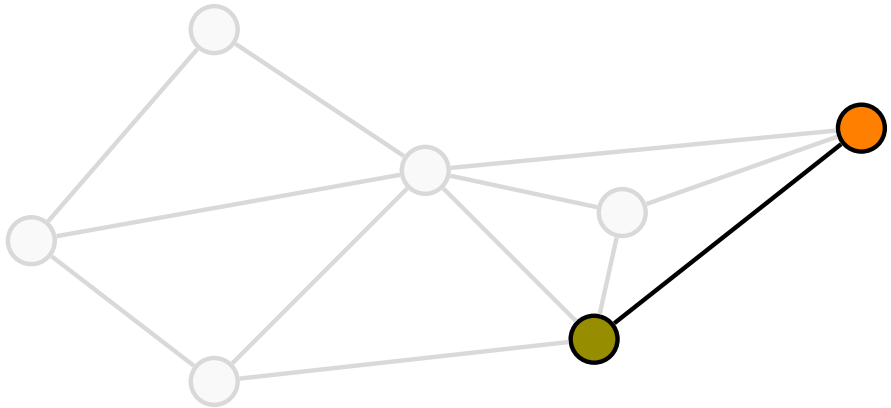

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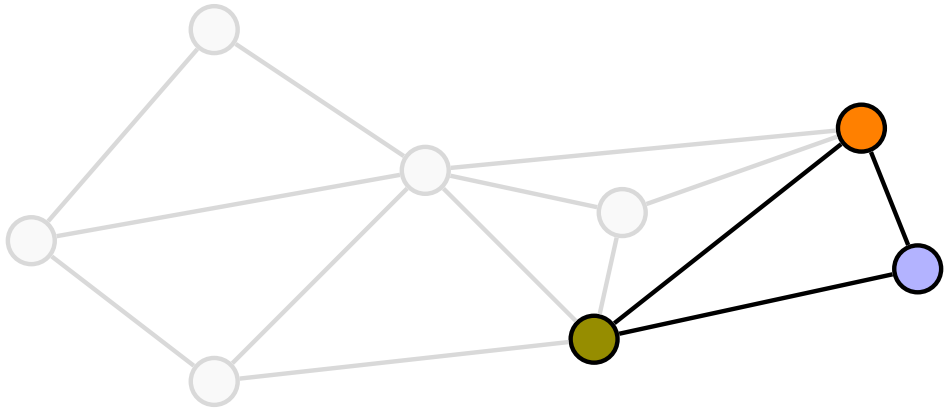

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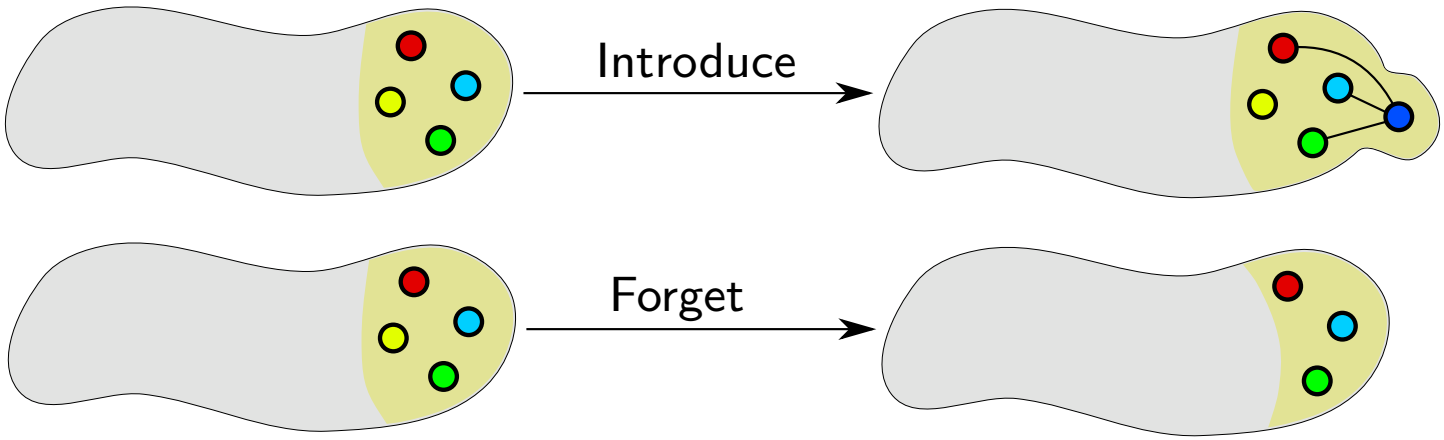

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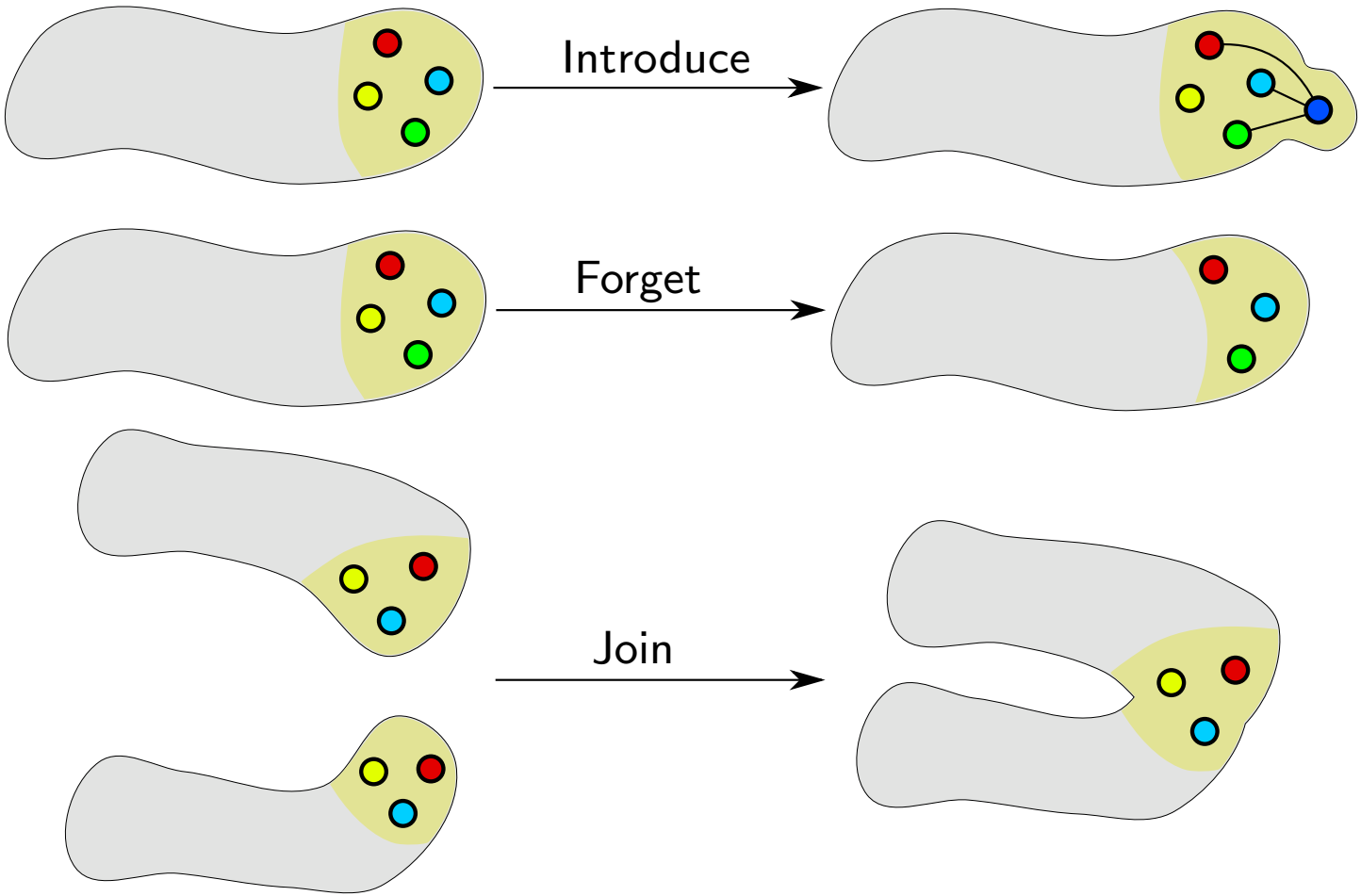

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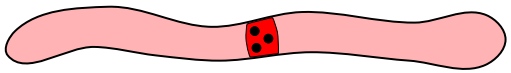

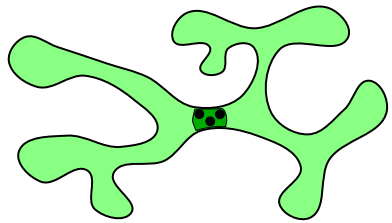

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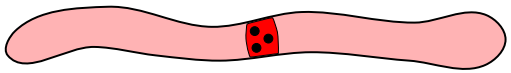

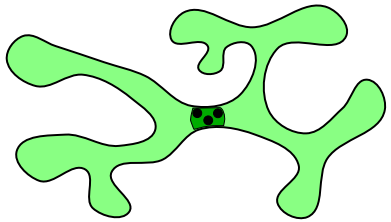

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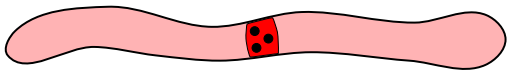

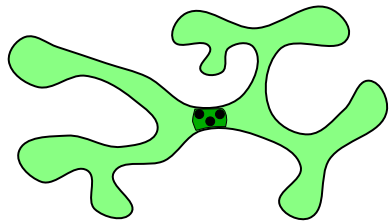

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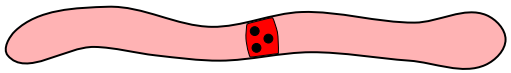

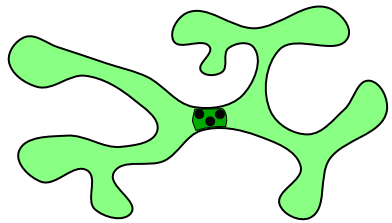

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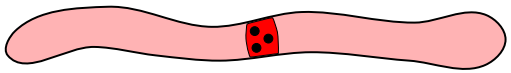

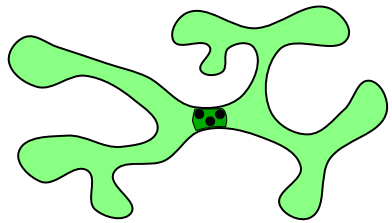

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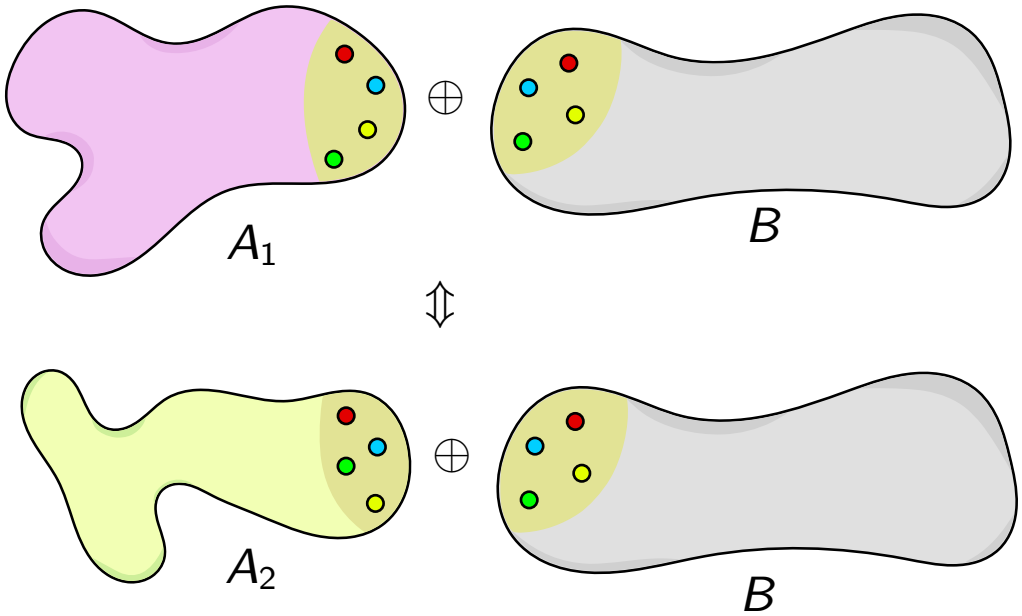
  - The parameter k is the *width* of the decomposition.

Recognizability

- Graph property Π is k-**recognizable** if the following Myhill-Nerode relation $\equiv_k$ over k-interface graphs has finite index.

$$A_1 \equiv_k A_2 \quad \Leftrightarrow \quad A_1 \oplus B \in \Pi \text{ iff } A_2 \oplus B \in \Pi \text{ for every } k\text{-interface graph } B.$$



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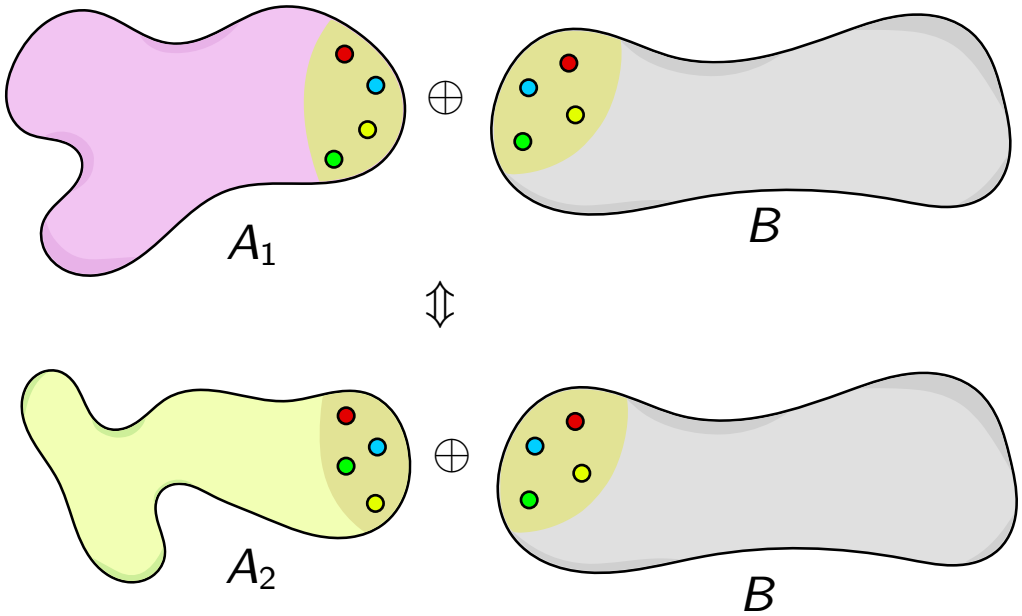

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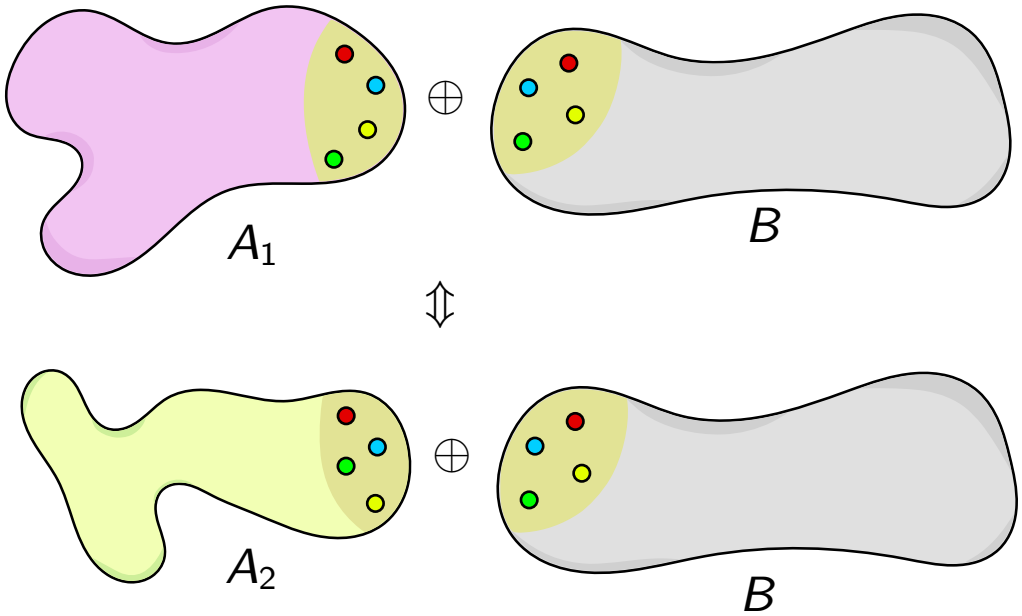

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- **Idea**: Recognizable properties can be verified using tree automata working on tree decompositions.

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Suppose

- Π is a recognizable graph property, and
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- (BP,16^+): The conjecture indeed holds.

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- **Thanks for attention!**