Hanf normal form for first-order logic with unary counting quantifiers

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In this talk

Hanf normal form characterises the locality of first-order logic (FO) on classes of structures of bounded degree.

- We generalise the notion of Hanf normal form to FO + sets **Q** of unary counting quantifiers

  Example: **D** consists of all modulo-counting quantifiers

  $$\varphi_{\text{EVEN}} \ := \ \exists^{0 \bmod 2} y \ y{=}y$$

- We provide a characterisation of the sets **Q** that permit generalised Hanf normal forms

- We show how to compute generalised Hanf normal forms effectively and in worst-case optimal time

Neighbourhoods and spheres

In this talk: All structures are finite, undirected and loop-free graphs.

(but all results hold for arbitrary finite relational signatures)

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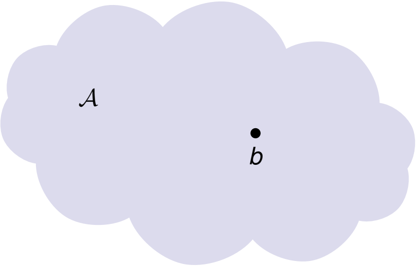

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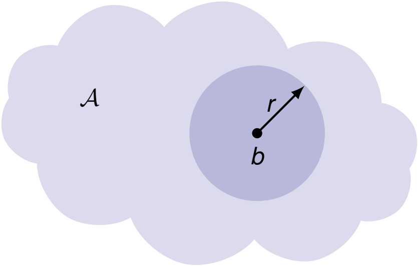

r-neighbourhood of b in $\mathcal{A}$

$$N_r^{\mathcal{A}}(b) \;:=\; \text{all nodes with distance} \leq r \text{ from } b$$

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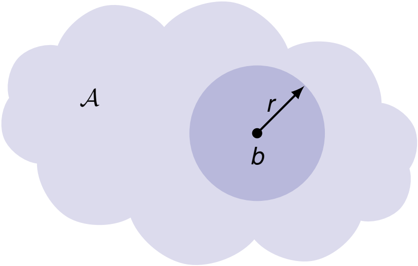

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r-sphere of b in $\mathcal{A}$

$$\mathcal{N}_r^{\mathcal{A}}(b) \;:=\; \big(\,\mathcal{A}[N_r^{\mathcal{A}}(b)], b\,\big)$$

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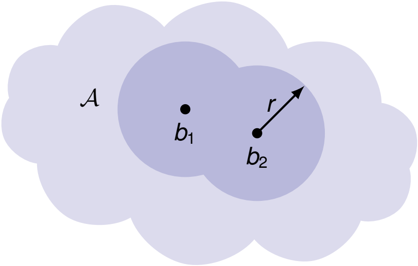

r-neighbourhood of $\overline{b}$ in $\mathcal{A}$

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Hanf's Theorem

For a graph $\mathcal{A}$ and an r-sphere τ, denote the number of realisations of τ in $\mathcal{A}$ by

$$\#_\tau(\mathcal{A}) \;:=\; |\{b \in A : \mathcal{N}_r^{\mathcal{A}}(b) \cong \tau\}|\,.$$

Theorem (Hanf 1965; Fagin, Stockmeyer, Vardi 1995)

For every degree bound $d \geq 0$ and each quantifier rank $q \geq 0$, there is

- *a radius $r \geq 0$ and*
- *a threshold $t \geq 0$,*

such that for all graphs $\mathcal{A}, \mathcal{B}$ with degree $\leq d$, the following holds:

If for every r-sphere τ with one centre,

$$\#_\tau(\mathcal{A}) \;=\; \#_\tau(\mathcal{B}) \qquad \text{or} \qquad \#_\tau(\mathcal{A}),\, \#_\tau(\mathcal{B}) \;\geq\; t\,,$$

then, for every FO-sentence φ with quantifier rank $\leq q$,

$$\mathcal{A} \models \varphi \quad \Longleftrightarrow \quad \mathcal{B} \models \varphi\,.$$

(degree $\leq d$: each node has $\leq d$ neighbours)

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Applications

- Inexpressibility results
- Algorithmic meta-theorems *(Seese 1996; Frick, Grohe 2004)*

Hanf normal form

Hanf normal form (for sentences)

Finite Boolean combination of counting-sentences

$$\exists^{\geq k} y \; \mathrm{sph}_\tau(y)$$

for $k \geq 1$ and spheres τ with one centre.

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Consequence of Hanf's Theorem

For each $d \geq 0$, every FO-sentence is d-equivalent to a Hanf normal form.

(d-equivalent: equivalent on all graphs of degree $\leq d$)

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There is an algorithm which, on input of

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For $d \geq 3$, this is worst-case optimal.

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for $k \geq 1$ and spheres τ with one centre and sphere-formulas

$$\mathrm{sph}_\rho(\overline{x})$$

for spheres ρ with $|\overline{x}|$ centres.

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$$(\mathcal{A}, \overline{a}) \models \mathrm{sph}_\rho(\overline{x}) \quad \iff \quad \mathcal{N}_r^{\mathcal{A}}(\overline{a}) \cong \rho$$

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Hanf normal form for FO + modulo-counting quantifiers

Modulo-counting quantifier: For a period $p \geq 2$,

$$\exists^{0 \bmod p} y \ \varphi(y)$$

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Consequence of Nurmonen's Theorem *(Nurmonen 2000)*

For each $d \geq 0$, every $\mathrm{FO}(\exists^{0 \bmod p})$-sentence is d-equivalent to a Hanf normal form.

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How can Hanf normal forms for $\mathrm{FO}(\exists^{0 \bmod p})$ be computed?

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Theorem

Let **D** denote the set of *all* modulo-counting quantifiers.
There is an algorithm which, on input of

- a degree bound $d \geq 0$ and
- an FO(**D**)-formula φ,

computes a d-equivalent Hanf normal form in time $2^{d^{2^{\mathcal{O}(\|\varphi\|)}}} \in$ 3-exp($\|\varphi\|$).

For $d \geq 3$, this is worst-case optimal.

Hanf normal form for FO + unary counting quantifiers

Unary counting quantifier: For every $S \subseteq \mathbb{N}$,

$$S y\ \varphi(y)$$

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Unary counting quantifier: For every $\mathsf{S} \subseteq \mathbb{N}$,

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For every set $\mathbf{Q} \subseteq \mathcal{P}(\mathbb{N})$, FO($\mathbf{Q}$) := extension of FO by all quantifiers $\mathsf{S} \in \mathbf{Q}$.

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Hanf normal form for FO($\mathbf{Q}$)

Finite Boolean combination of sphere-formulas and counting-sentences of the shape

$$(\mathrm{S}{+}k)y\ \mathrm{sph}_{\tau}(y)$$

where $\mathrm{S} \in \mathbf{Q}$ and $(\mathrm{S}{+}k) := \{n + k : n \in \mathrm{S}\}$.

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For every set $\mathbf{Q} \subseteq \mathcal{P}(\mathbb{N})$, $\mathrm{FO}(\mathbf{Q})$:= extension of FO by all quantifiers $\mathsf{S} \in \mathbf{Q}$.

Hanf normal form for FO(Q)

Finite Boolean combination of sphere-formulas and counting-sentences of the shape

$$\mathcal{A} \models (\mathsf{S}{+}k)y\ \mathrm{sph}_\tau(y) \iff \#_\tau(\mathcal{A}) - k \in \mathsf{S}$$

where $\mathsf{S} \in \mathbf{Q}$ and $(\mathsf{S}{+}k) := \{n + k : n \in \mathsf{S}\}$.

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$\mathbf{Q} \subseteq \mathcal{P}(\mathbb{N})$ permits Hanf normal form

iff for each $d \geq 0$ and every FO($\mathbf{Q}$)-formula,
there is a d-equivalent FO($\mathbf{Q}$)-formula in Hanf normal form

Examples: $\emptyset$ and $\mathbf{D}$ permit Hanf normal form.

Results

Which sets $\mathbf{Q} \subseteq \mathcal{P}(\mathbb{N})$ permit Hanf normal form?

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Theorem

For every set $\mathbf{Q} \subseteq \mathcal{P}(\mathbb{N})$ *of unary counting quantifiers,*

$$\mathbf{Q} \textit{ permits Hanf normal form} \quad \Longrightarrow \quad \textit{all } \mathsf{S} \in \mathbf{Q} \textit{ are "ultimately periodic".}$$

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How to compute such Hanf normal forms?

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For every set $\mathbf{Q} \subseteq \mathcal{P}(\mathbb{N})$ *of unary counting quantifiers,*

$\mathbf{Q}$ *permits Hanf normal form* $\iff$ *all* $S \in \mathbf{Q}$ *are "ultimately periodic".*

Theorem

There is an algorithm which, on input of

- a degree bound $d \geq 0$ and
- an FO($\mathbf{Q}$)-formula φ (where all $S \in \mathbf{Q}$ are "ultimately periodic"),

computes a d-equivalent Hanf normal form in time $2^{d^{2^{\mathcal{O}(\|\varphi\|)}}} \in$ 3-exp($\|\varphi\|$).

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An application:

Generalisation of Seese's model-checking algorithm for graphs of bounded degree

If all $S \in \mathbf{Q}$ are "ultimately periodic", then, for each $d \geq 0$,
model-checking for FO($\mathbf{Q}$) on graphs of degree $\leq d$ is fixed-parameter tractable.

Ultimately periodic quantifiers

$S \subseteq \mathbb{N}$ is ultimately periodic

iff there is an offset $n_0 \geq 0$ and a period $p \geq 1$ such that

$$\text{for all } n \geq n_0, \qquad n \in S \quad \Longleftrightarrow \quad n + p \in S\,.$$

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Examples

- $\exists$ and $\exists^{0 \bmod p}$ are ultimately periodic,
- $\{n^2 : n \in \mathbb{N}\}$ and $\{n : n \text{ is prime}\}$ are not ultimately periodic.

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Only ultimately periodic quantifiers permit Hanf normal form

Let $\sigma := \{P\}$ and let $\mathbf{Q} \subseteq \mathcal{P}(\mathbb{N})$ with $\mathsf{S} \in \mathbf{Q}$ not ultimately periodic.
There is no Hanf normal form $\delta \in \mathrm{FO}(\mathbf{Q})[\sigma]$ such that

$$\text{for every } \sigma\text{-structures } \mathcal{A}, \quad \mathcal{A} \models \delta \quad \Longleftrightarrow \quad |A| \in \mathsf{S}.$$

Consequence: There is no Hanf normal form for $\mathsf{S}y\; y{=}y$.

Thank you very much for your attention and for the slide reviews!

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