Mealy Automata, Orbit Trees and the Burnside Problem

Thibault Godin

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ANR JCJC 12 JS02 012 01

The Burnside problem

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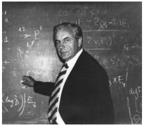

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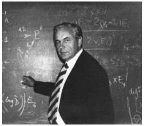

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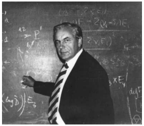

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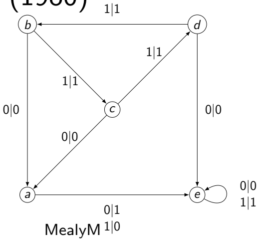
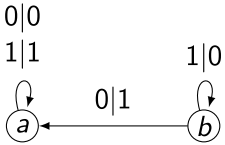


How to Generate Automaton Groups



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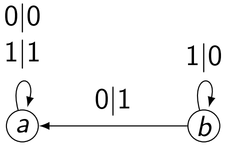

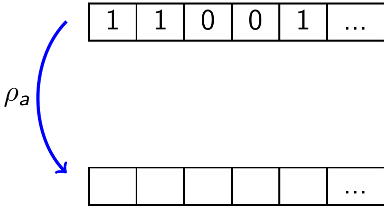

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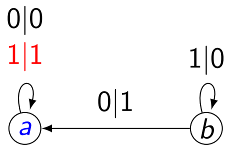

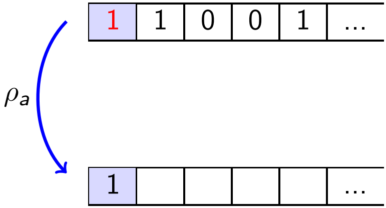

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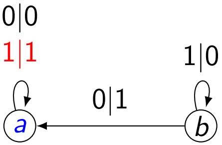

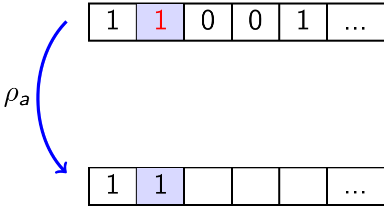

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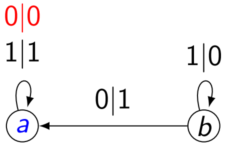

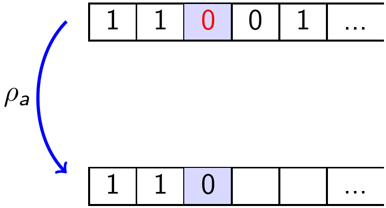

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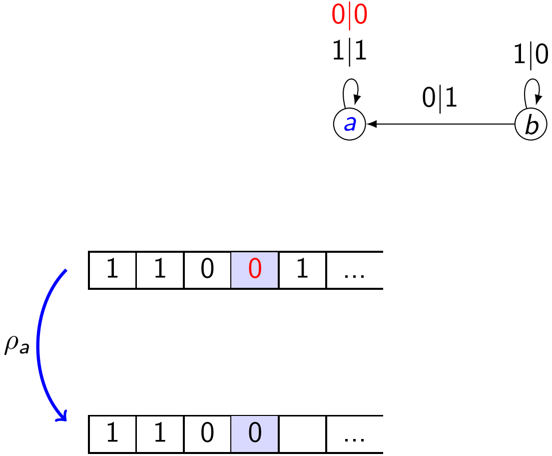

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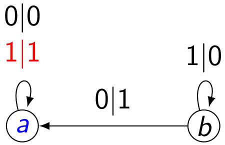

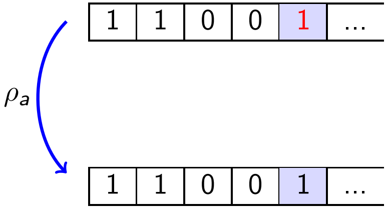

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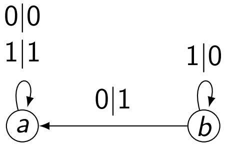

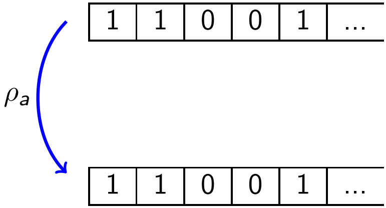

$$\rho_a : \Sigma^* \to \Sigma^*$$

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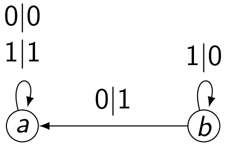

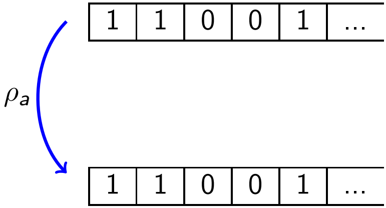

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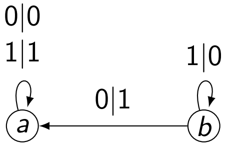

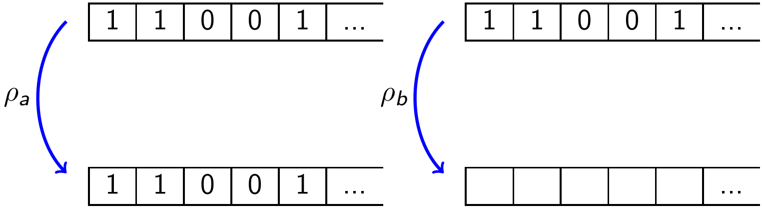

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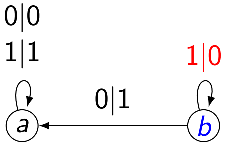

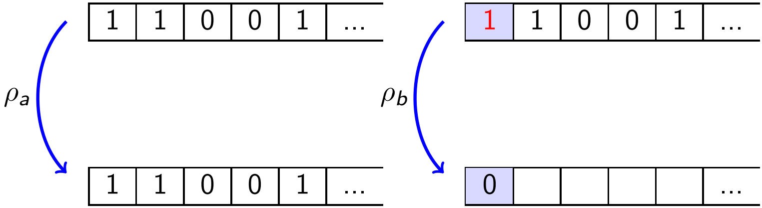

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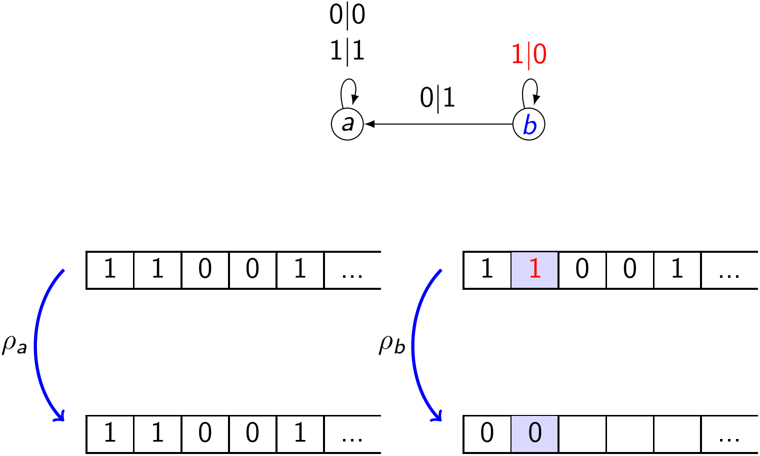

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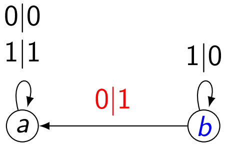

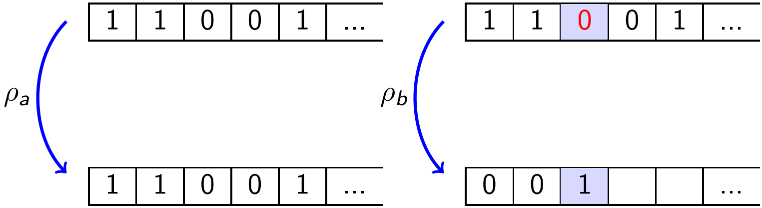

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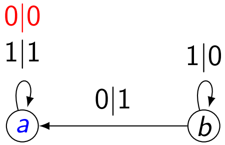

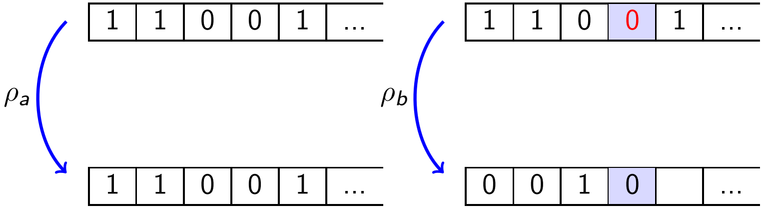

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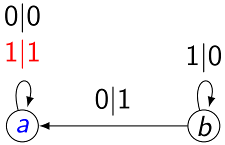

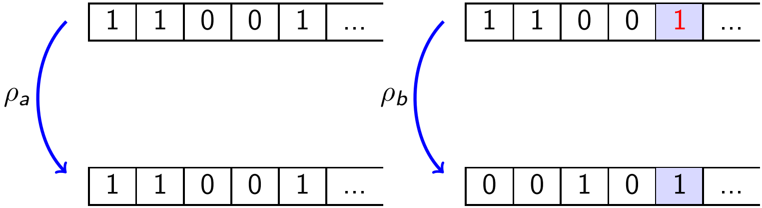

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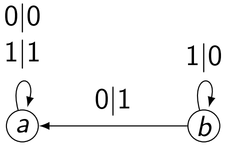

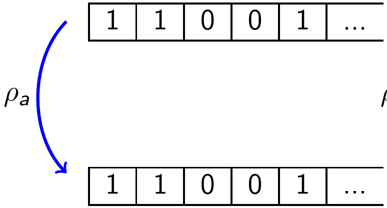

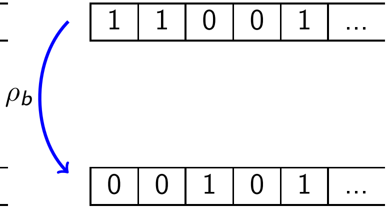

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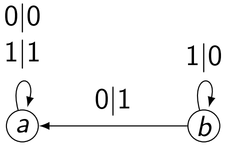

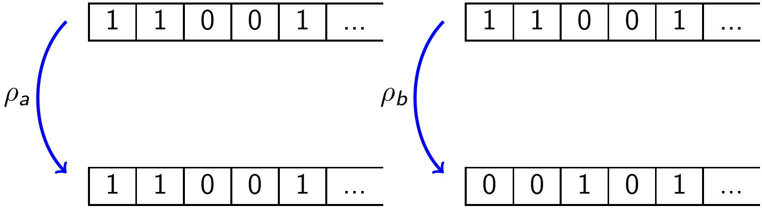

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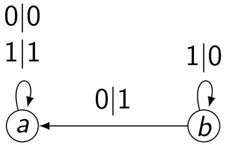

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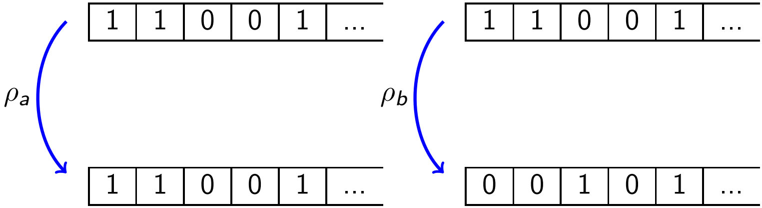

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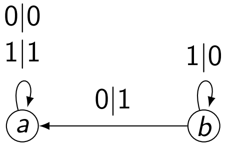

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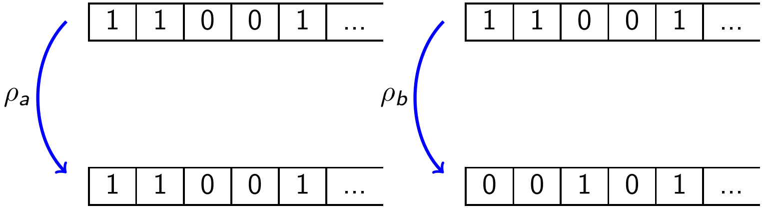

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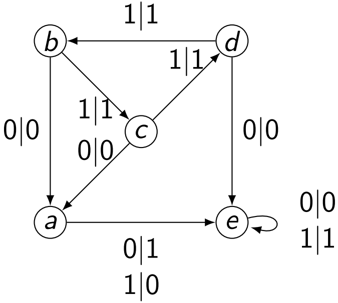
About the Grigorchuk automaton



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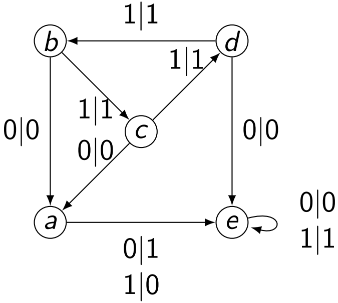

Action of a state on the letters:

$\rho_a : 0 \mapsto 1 \mapsto 0$

$\rho_b : 0 \mapsto 0; \quad 1 \mapsto 1$

$\rho_c : 0 \mapsto 0; \quad 1 \mapsto 1$

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$\rightarrow$permutations

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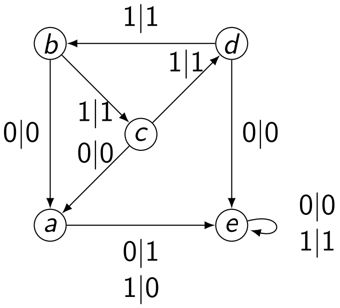

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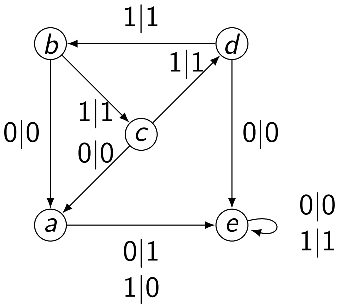

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→permutations

→ **invertible**

Action of a letter on the states:

$\delta_0 : a, d, e \mapsto e; \quad b, c \mapsto a$

→not a permutation

→ **non-reversible**

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A 2-state invertible-reversible Mealy automaton cannot generate an infinite Burnside group

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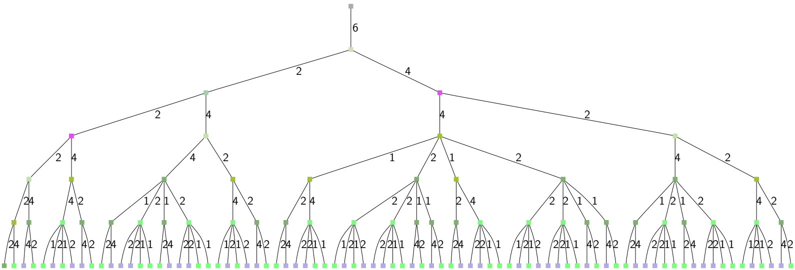

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Theorem 3 [G., Klimann and Picantin, LATA '15]

An invertible-reversible Mealy automaton with a non coreversible component cannot generate an infinite Burnside group

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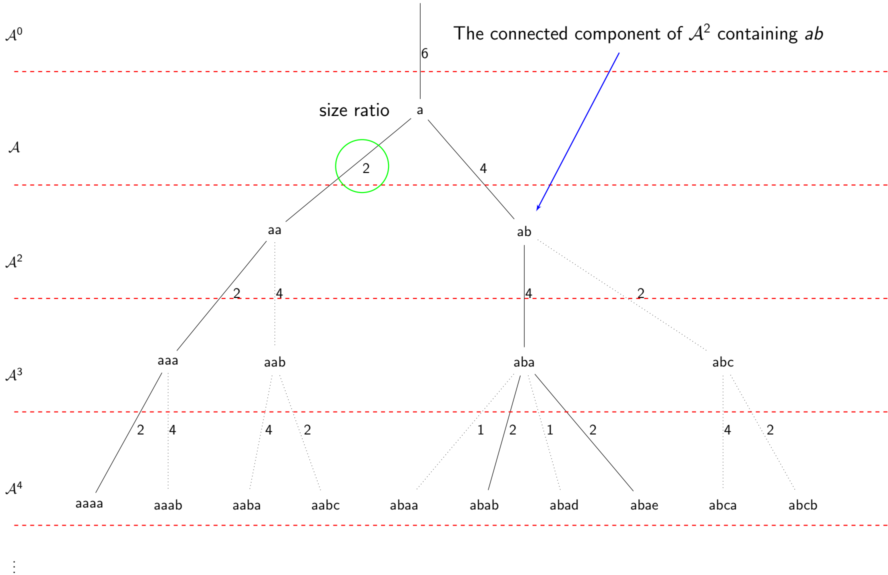
Theorem 4 [G., Klimann MFCS'16]

A connected invertible-reversible Mealy automaton with a prime number of states cannot generate an infinite Burnside group.

Orbit Tree



$\mathcal{A}^0$
$\mathcal{A}$
$\mathcal{A}^2$
$\mathcal{A}^3$
$\mathcal{A}^4$
⋮
6
The connected component of $\mathcal{A}^2$ containing *ab*
size ratio
a
2
4
aa
ab
2
4
4
2
aaa
aab
aba
abc
2
4
4
2
1
2
1
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aaaa
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aabc
abaa
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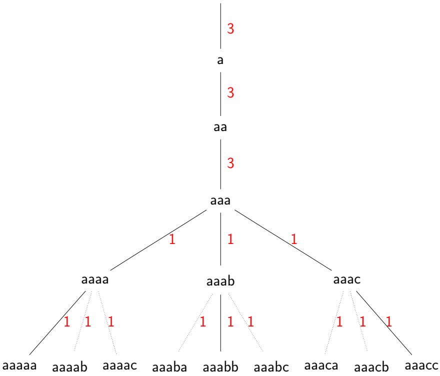
abad
abae
abca
abcb

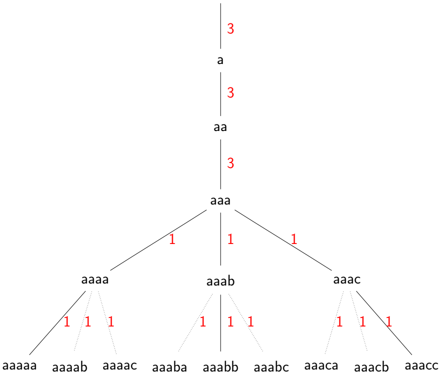
Boundedness



Boundedness

Prop

$\langle \mathcal{A} \rangle$ is finite iff the sizes of the connected components of $(\mathcal{A}^n)_n$ are bounded



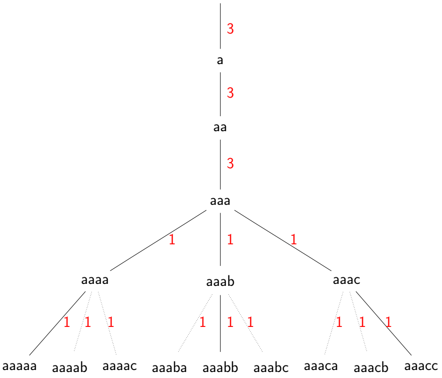
Boundedness

Prop

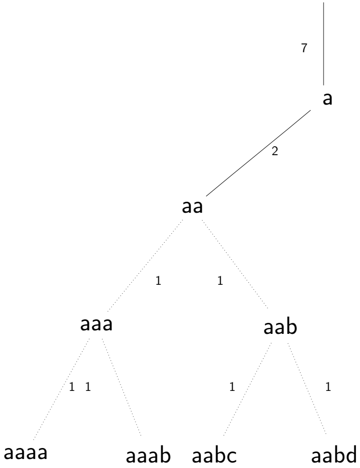
$\langle \mathcal{A} \rangle$ is finite iff the sizes of the connected components of $(\mathcal{A}^n)_n$ are bounded

Prop

ρ_u has finite order iff the sizes of the connected components of $(\mathcal{A}^n)_n$ containing u^n are bounded



Jungle tree



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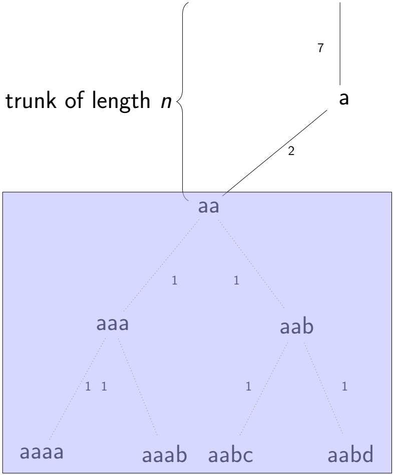

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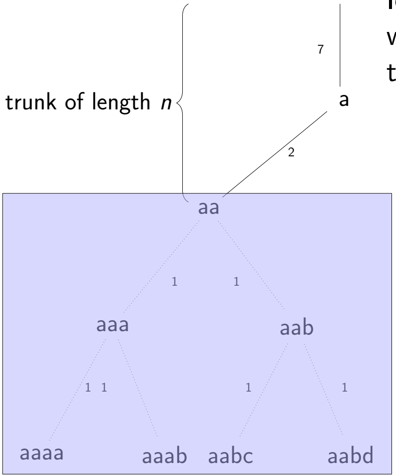

Idea: for any word, find a word with the same action in the jungle-tree

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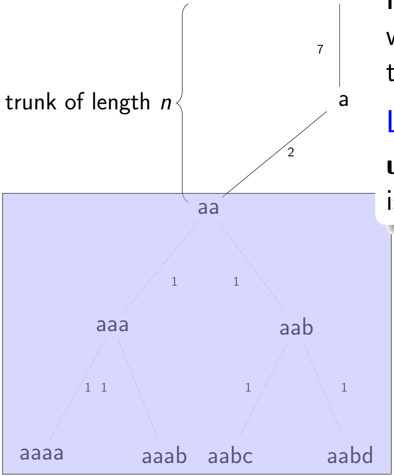

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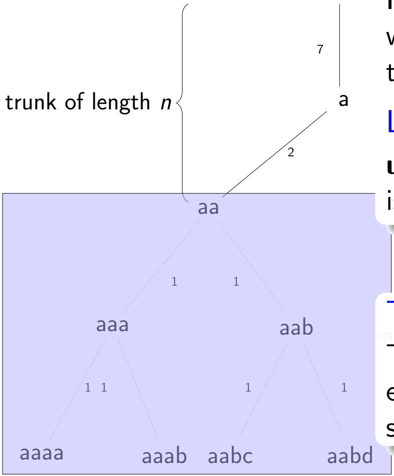
Lemma

$\mathbf{u} \sim \mathbf{v} \Leftrightarrow \mathbf{u}w\mathbf{v} \in \mathcal{J}$ and $\rho_{\mathbf{u}w} = id$ is an equivalence relation on Q^n

Jungle tree



Idea: for any word, find a word with the same action in the jungle-tree

Lemma

$\mathbf{u} \sim \mathbf{v} \Leftrightarrow \mathbf{u}w\mathbf{v} \in \mathcal{J}$ and $\rho_{\mathbf{u}w} = id$ is an equivalence relation on Q^n

Theorem

The set of first letter of an equivalence class is the whole stateset

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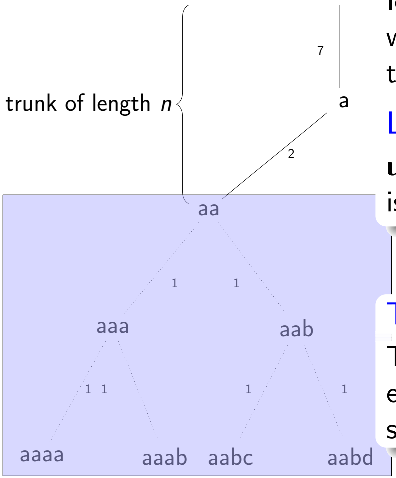

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$$u_1 u_2 \ldots \equiv \mathbf{v} w u_1$$

with $\rho_{\mathbf{v}w} = id$

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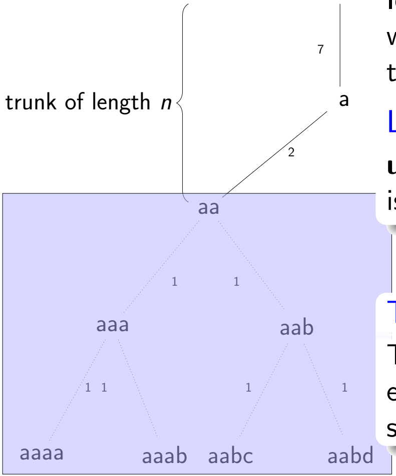

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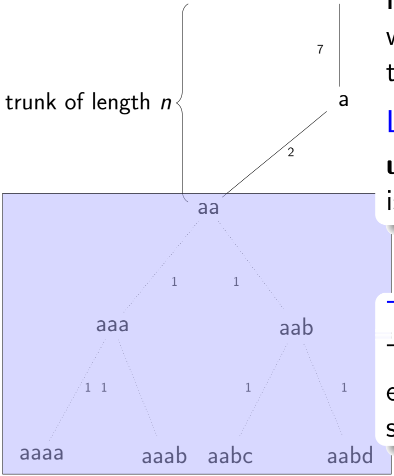

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Theorem

The set of first letter of an equivalence class is the whole stateset

$$u_1 u_2 \ldots \equiv \mathbf{v} w u_1 \mathbf{v'} w' u2 \ldots$$

with $\rho_{\mathbf{v}w} = id$ and $\rho_{\mathbf{v'}w'} = id$

Theorem [G., Klimann 2016]

A connected invertible-reversible Mealy automaton with a prime number of states does not generate an infinite Burnside group.

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A connected invertible-reversible Mealy automaton with a prime number of states does not generate an infinite Burnside group.

- Extend the result to any connected automaton
- Find an approach for non connected automata
- Decide the finiteness for automaton groups
- Find other structural properties

Thanks