Undecidability of Two-dimensional Robot Games

Reino Niskanen[1] Igor Potapov[1], Julien Reichert[2]

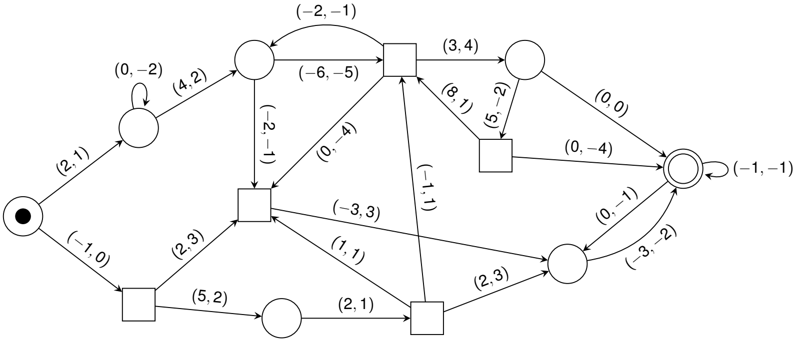
[1]Department of Computer Science
University of Liverpool, UK

[2]LSV, ENS Cachan, France

Highlights of Logic, Games and Automata

Introduction

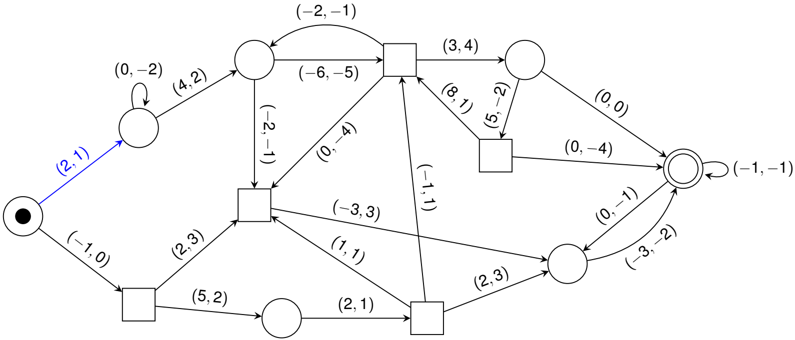
Counter reachability games



Counter values

$(3, 3)$

Counter reachability games



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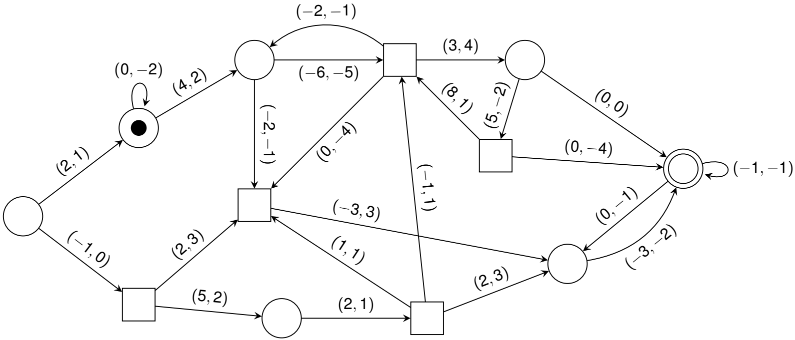

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$(3, 3) \to (5, 4)$

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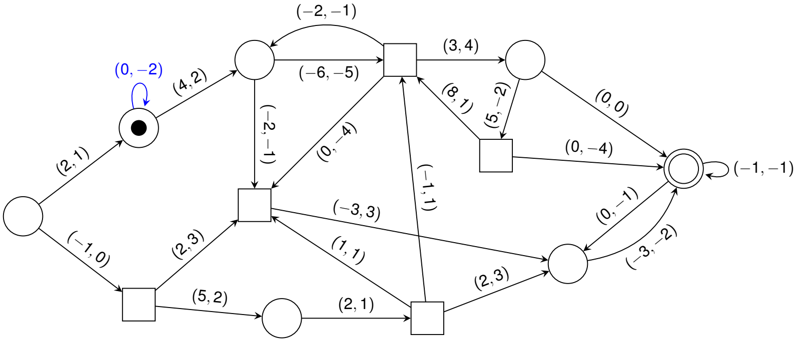

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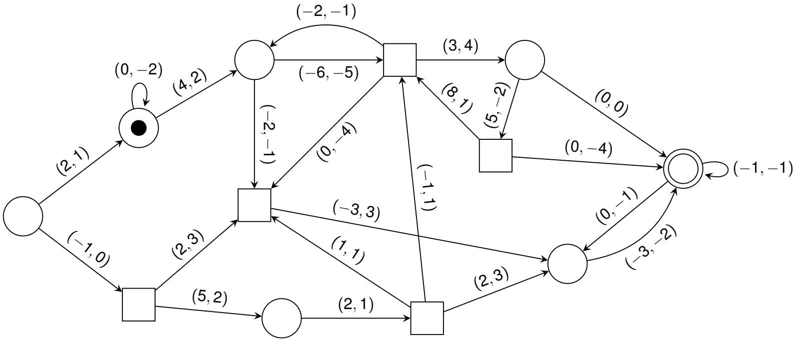

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$$(3, 3) \to (5, 4) \to (5, 2)$$

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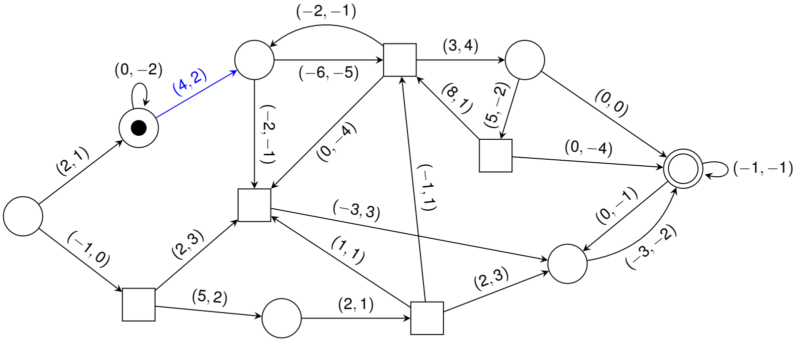

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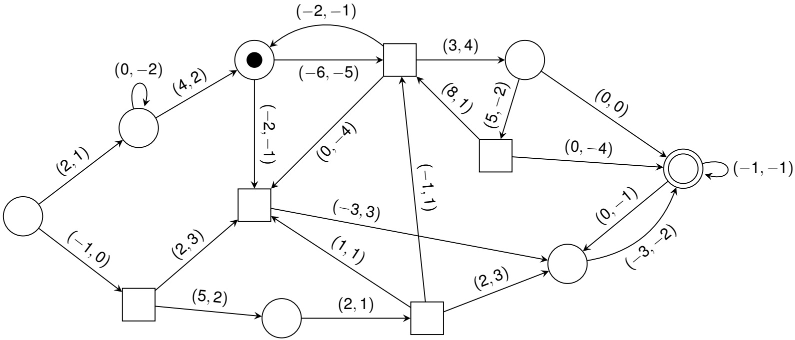

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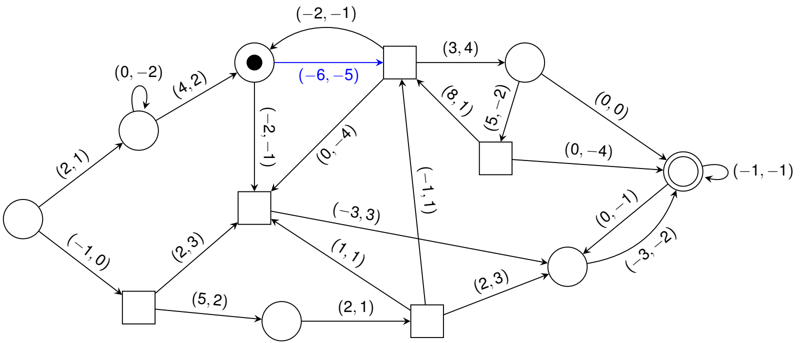

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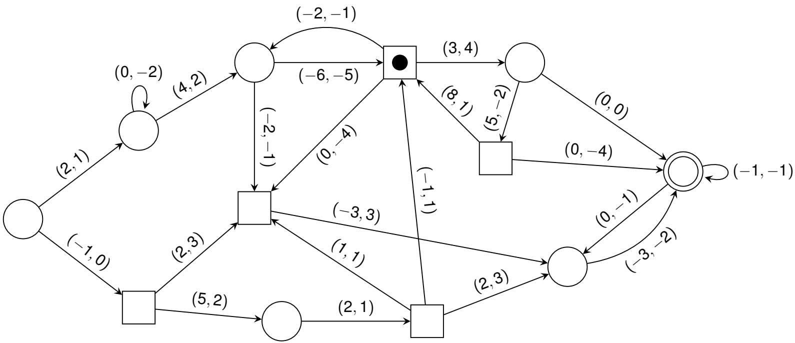

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$(3, 3) \to (5, 4) \to (5, 2) \to (9, 4) \to (3, -1)$

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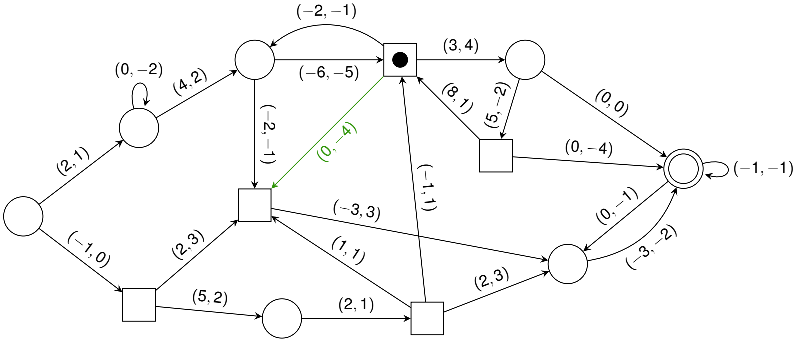

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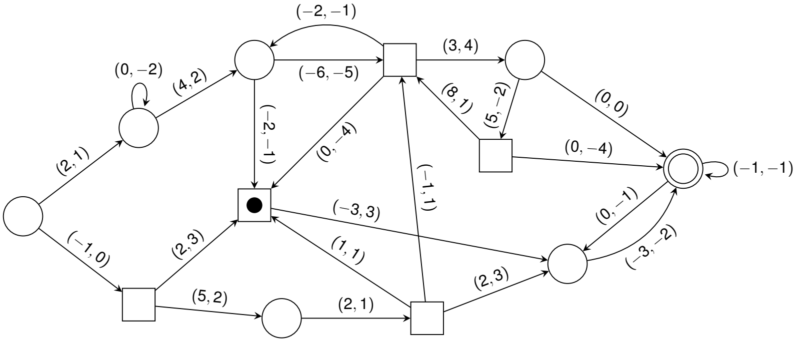

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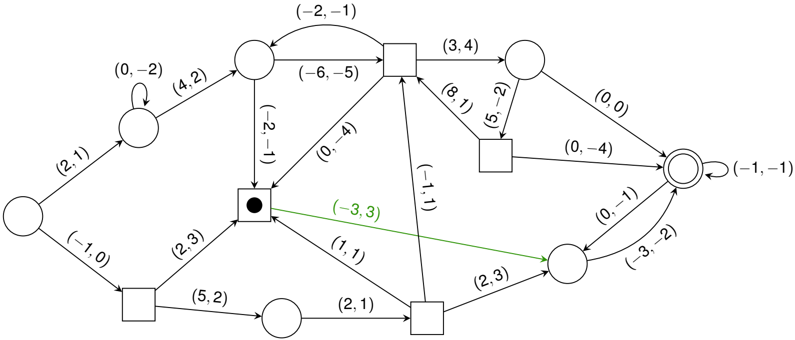

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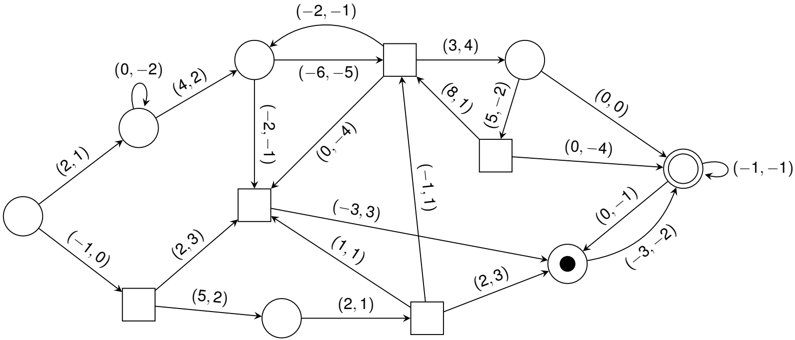

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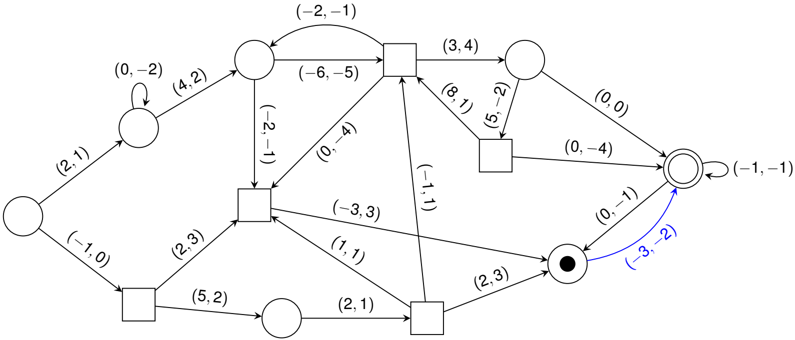

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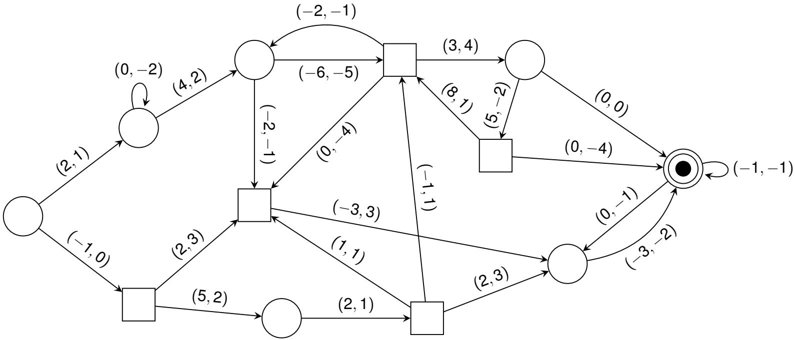

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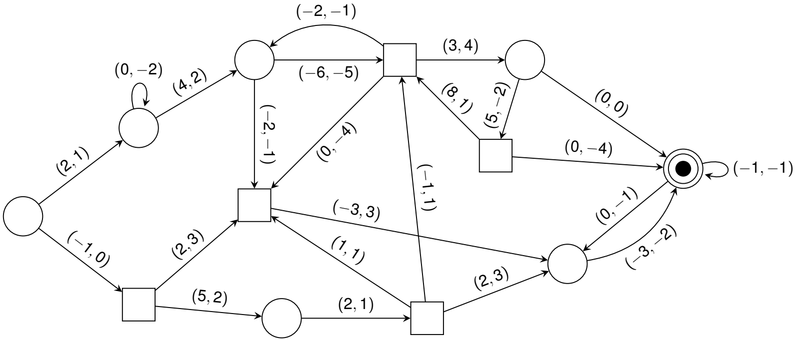
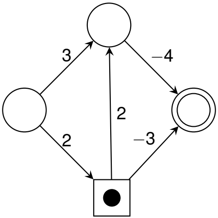


Counter values

$(3,3) \to (5,4) \to (5,2) \to (9,4) \to (3,-1) \to (3,-5) \to (6,-2) \to (3,-4) \to \cdots$

Counter reachability games

- Played on a labeled directed graph $G = (V, E)$ with edges labeled by $\mathbf{x} \in \mathbb{Z}^n$.
- Two players: Eve ($\bigcirc$) , Adam ($\square$).
- A configuration $[v, \mathbf{z}] \in V \times \mathbb{Z}^n$.
- A successor configuration is $[v', \mathbf{z} + \mathbf{z}']$, where $[v, \mathbf{z}', v'] \in E$ and the owner of v chose it.
- The initial and target configurations.
- Eve has a winning strategy if the target configuration is reachable for every choice of Adam.



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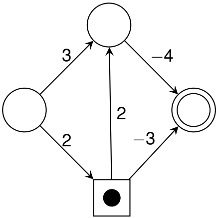


The decision problem

Given a graph G, configurations $[v_0, \mathbf{z}_0]$ and $[v_f, \mathbf{z_f}]$. Does there exist a winning strategy for Eve to reach $[v_f, \mathbf{z_f}]$ from $[v_0, \mathbf{z}_0]$?

Counter reachability games

The decision problem

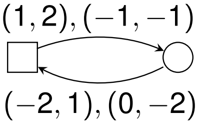
Given a graph $G = (V, E)$*, initial and target configurations,* $[v_0, \mathbf{z}_0]$ *and* $[v_f, (0, \ldots, 0)]$*. Does there exist a winning strategy for Eve to reach* $[v_f, (0, \ldots, 0)]$ *from* $[v_0, \mathbf{z}_0]$*?*

Known for counter reachability games:

One-dimensional	**EXPSPACE**-complete
Two-dimensional	Undecidable

Robot games

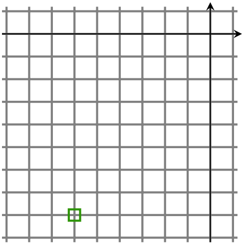
What if we have a simpler graph?



- Proposed by Doyen and Rabinovich in 2011. Claimed to be undecidable in dimension 9.
- **EXPTIME**-complete in dimension one [Arul, Reichert, QAPL 2013].
- Undecidable in dimension three [Reichert, PhD thesis 2015].
- Remained open in dimension two.

Another way to look at robot games

- Played on integer lattice $\mathbb{Z}^n$.
- Adam and Eve move a token on the lattice.
- Eve's goal is to reach $(0, \ldots, 0)$. Adam's goal is to avoid it.



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Eve's moves: $\{(2, 2), (1, 4)\}$

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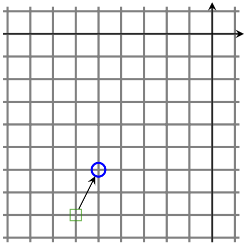

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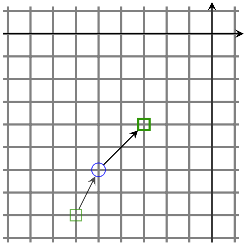

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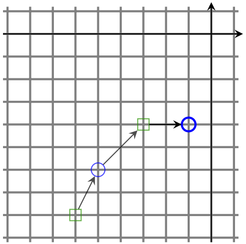

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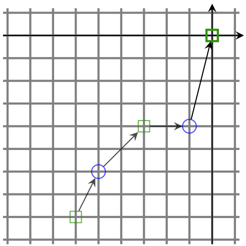

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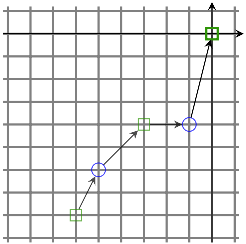

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Theorem

Given moves of Adam and Eve, $A, E \subseteq \mathbb{Z}^2$, an initial vector $\mathbf{x} \in \mathbb{Z}^2$. It is undecidable whether Eve has a winning strategy.

The basis of undecidability proofs

Marvin Minsky
(1927-2016)

Deterministic two-counter Minsky machine (2CM):

- Two counters, c_1 and c_2.
- m instructions: $1 : \textsc{Ins}_1, \ldots, m : \textsc{Ins}_m$, where $\textsc{Ins}_i$ is
  - i: `c1++; goto` k, or
  - i: `c2++; goto` k, or
  - i: `if c1=0 goto` k `else c1--; goto` j, or
  - i: `if c2=0 goto` k `else c2--; goto` j, or
  - i: `halt`.
- The halting problem is undecidable [Minsky, 1967].

Robot games

Two counter machines to robot games

- Eve simulates transitions of the machine.
- Adam verifies that zero-checks are performed correctly.
- Adam verifies that Eve moves according to the state structure.

The main challenge is to encode the state structure into integers.

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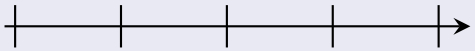

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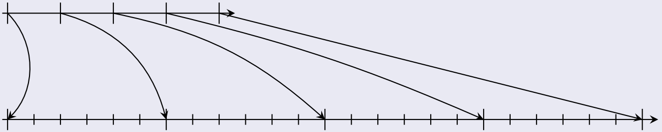

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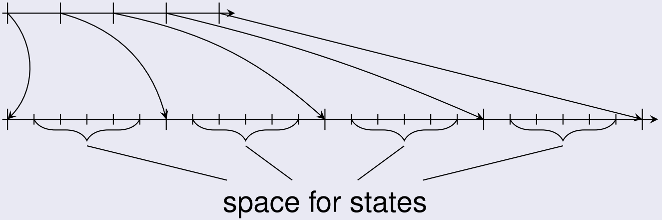
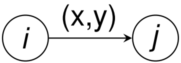


Removing the states

A move in 2CM:



The corresponding move in 2RG:

$(x, yN - 2^i + 2^j)$, where $N \in \mathbb{N}$.

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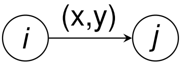

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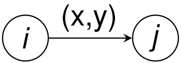
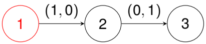


The corresponding move in 2RG:

$(x, yN - 2^i + 2^j)$, where $N \in \mathbb{N}$.

- Too simple: Several wrong moves can result in a right one.
- Even more space is needed to make sure that wrong moves do not affect the rest of the computation.

Removing the states



$$\overbrace{(1, 0 \cdot 4 \cdot 8^{10}}^{\text{2CM counters}} + \overbrace{0 \cdot 8^{9} + 0 \cdot 8^{8} + 0 \cdot 8^{7} + 0 \cdot 8^{6} + 0 \cdot 8^{5} + 0 \cdot 8^{4}}^{\text{emptying states}} + \overbrace{0 \cdot 8^{3} + 0 \cdot 8^{2} + 1 \cdot 8^{1}}^{\text{states of 2CM}} \overbrace{-1 \cdot 8^{0}}^{\top_{00}})$$

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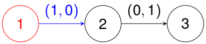

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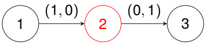

Encoded in such way that Eve has to ensure that only one coefficient is non-zero.

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$$\downarrow (1, -8^{1} + 8^{2})$$

$$(2, 0 \cdot 4 \cdot 8^{10} + 0 \cdot 8^{9} + 0 \cdot 8^{8} + 0 \cdot 8^{7} + 0 \cdot 8^{6} + 0 \cdot 8^{5} + 0 \cdot 8^{4} + 0 \cdot 8^{3} + \boxed{1 \cdot 8^{2}} + \boxed{0 \cdot 8^{1}} - 1 \cdot 8^{0})$$

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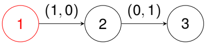

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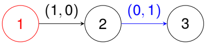

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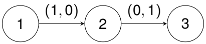

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$$\downarrow (0, 1\cdot 4\cdot 8^{10} - 8^{2} + 8^{3})$$

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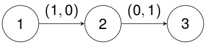

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- Ensures that carries cannot be abused (essentially by making the distance between two correct configurations large enough)
- Additional tricks to ensure that zero-checks are performed correctly.

Conclusion

Summary

Theorem

Given moves of Adam and Eve, $A, E \subseteq \mathbb{Z}^2$, an initial vector $\mathbf{x} \in \mathbb{Z}^2$. It is undecidable whether Eve has a winning strategy.

Game	Dimension		
	1	2	≥ 3
counter reachability games	**EXPSPACE**-complete	U	–
robot games	**EXPTIME**-complete	?	U

Summary

Theorem

Given moves of Adam and Eve, $A, E \subseteq \mathbb{Z}^2$, an initial vector $\mathbf{x} \in \mathbb{Z}^2$. It is undecidable whether Eve has a winning strategy.

Game	Dimension		
	1	2	≥ 3
counter reachability games	**EXPSPACE**-complete	U	–
robot games	**EXPTIME**-complete	U	–

Thank you for your attention!