Order Invariance on Decomposable Structures

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Which graph properties can be expressed in logics?

Monadic second-order logic (MSO)

Syntax φ uses $(x, y) \in E$ and quantifiers $\exists x \in V$, $\exists X \subseteq V$.

Semantics $G \models \varphi$ means φ holds in $G = (V, E)$.

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Modulo-counting MSO (CMSO)

Syntax MSO-formula φ also uses $\text{MOD}_2(X)$, $\text{MOD}_3(X)$, ...

Semantics $G \models \text{MOD}_m(S)$ means $|S|$ is a multiple of m.

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Order-invariant MSO (<-inv-MSO)

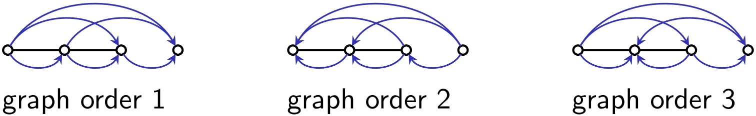
Syntax MSO-formula φ also uses an order $x < y$,
which is not part of the structure

Semantics ?

Order-invariant formulas

graph

Order-invariant formulas



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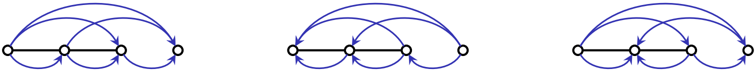
Order-invariant formulas



graph order 1 graph order 2 graph order 3

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- Formula φ is order-invariant if it accesses an order, but model relation is independent of the concrete order:
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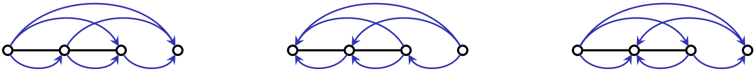

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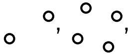
Example: Even-sized sets are definable in $<$-inv-MSO

$\textsc{even-set} = \{$  $, \ldots \} \in$ $<$-inv-MSO using order as path

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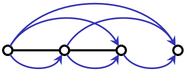

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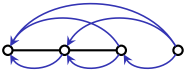
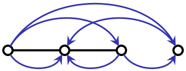


graph order 2



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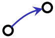
$\textsc{even-set} = \{$ , , $\ldots \} \in$ $<$-inv-MSO using order as path

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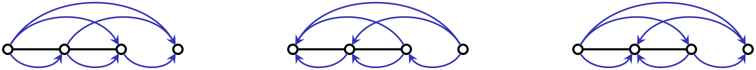

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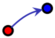
$\textsc{even-set} = \{$ , , $\ldots \} \in$ $<$-inv-MSO using order as path

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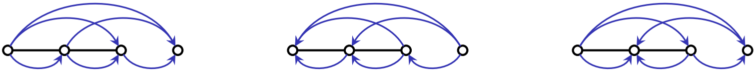

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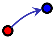
$\textsc{even-set} = \{$ , , $\ldots \} \in$ $<$-inv-MSO using order as path

Fact

MSO $\subsetneq$ CMSO $\subsetneq$ $<$-inv-MSO [Ganzow and Rubin, 2008]

Order invariance on “structured” graphs collapses.

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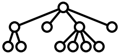
$\mathrm{CMSO} = {<}$-inv-MSO *on trees* 

[Courcelle, 1996]

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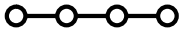
MSO $=$ $<$-inv-MSO *on paths* 

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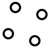
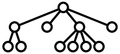
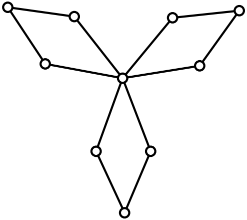
CMSO $=$ $<$-inv-MSO *on sets* 

Fact

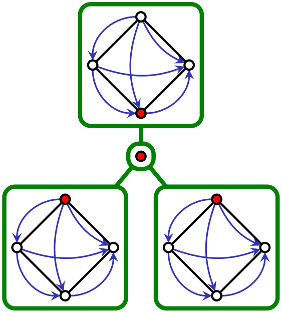
CMSO $=$ $<$-inv-MSO *on trees*  [Courcelle, 1996]

Can we extend this result to tree decompositions?

Tree decompositions allow combination of bag orders.



graph



tree decomposition with ordered bags

Main contribution: Lifting Theorem

CMSO = <-inv-MSO *on graphs with tree decompositions having*

- *ordered bags, and*
- *size-bounded separators.*

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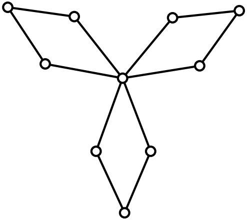

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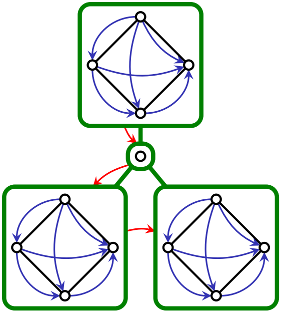

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Results and Outlook

Theorem

$\textsc{cmso} = <\text{-inv-}\textsc{mso}$ *on graphs of bounded tree width.*

Plan of the proof.

1. MSO-define decompositions along clique separators.
2. MSO-define orderings for clique-separator-free bags.
3. Lift-up definability along the decomposition. □

Alternative proof by [Bojańczyk and Pilipczuk, 2016]

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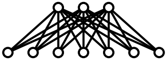
CMSO $=$ $<$-inv-MSO *on graphs without* $K_{3,\ell} =$  *as a minor (e.g. planar, bounded genus graphs).*

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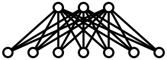
$\mathrm{CMSO} = {<}\text{-inv-}\mathrm{MSO}$ *on graphs without* $K_{3,\ell} =$  *as a minor (e.g. planar, bounded genus graphs).*

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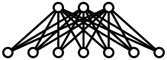
$\textsc{CMSO} = {<}\text{-inv-}\textsc{MSO}$ *on graphs without* $K_{3,\ell} =$  *as a minor (e.g. planar, bounded genus graphs).*

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Outlook

- Possible to extend to graphs without $K_\ell =$  as a minor?
- Since ${<}\text{-inv-}\textsc{FO} \subseteq \textsc{MSO}$, could it be that $\textsc{FO} = {<}\text{-inv-}\textsc{FO}$?

References

Bojańczyk, M. and Pilipczuk, M. (2016).
Definability equals recognizability for graphs of bounded tree width.
In Proceedings of LICS 2016. IEEE Computer Society.
to appear.

Courcelle, B. (1996).
The monadic second-order logic of graphs X: Linear orderings.
Theoretical Computer Science, 160(1-2):87–143.

Ganzow, T. and Rubin, S. (2008).
Order-invariant MSO is stronger than counting MSO in the finite.
In Proceedings of 25th Annual Symposium on Theoretical Aspects of Computer Science (STACS 2008), pages 313–324.