# Easy to Win, Hard to Master:
## Optimal Strategies in Parity Games with Costs

Joint work with Martin Zimmermann

Alexander Weinert

Saarland University

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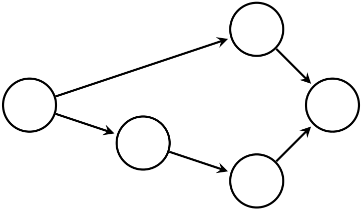
Highlights 2016 - Brussels

# Parity Games



Example due to (Fijalkow and Chatterjee, Infinite-state games, 2013)

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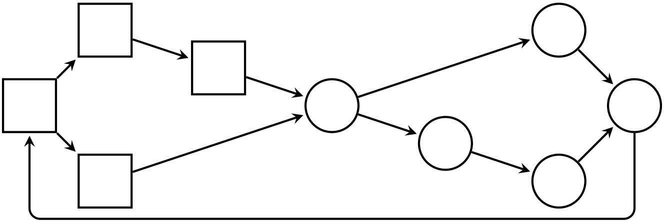

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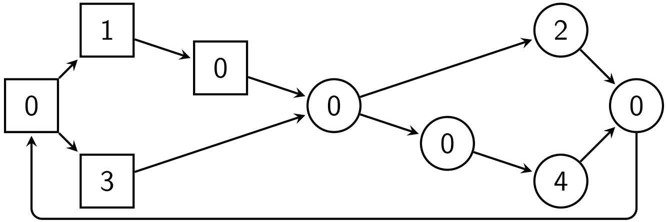


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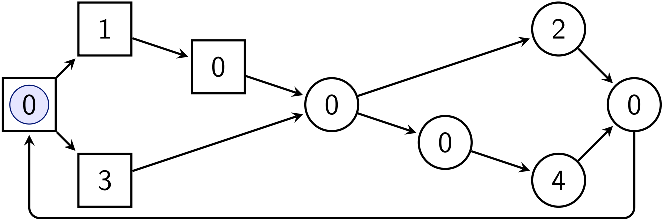


0

Example due to (Fijalkow and Chatterjee, Infinite-state games, 2013)

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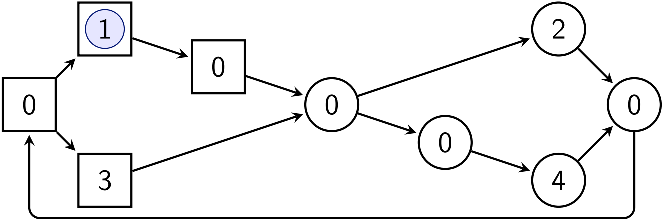


$0 \longrightarrow 1$

Example due to (Fijalkow and Chatterjee, Infinite-state games, 2013)

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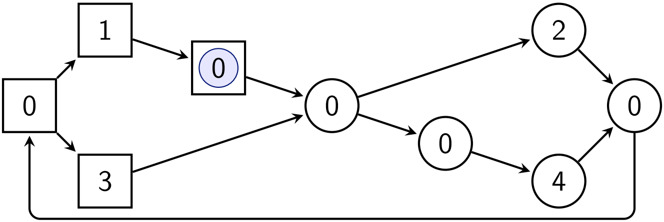


$0 \longrightarrow 1 \longrightarrow 0$

Example due to (Fijalkow and Chatterjee, Infinite-state games, 2013)

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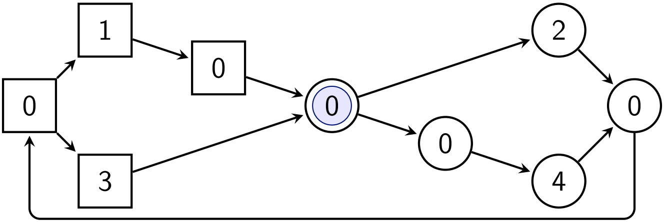


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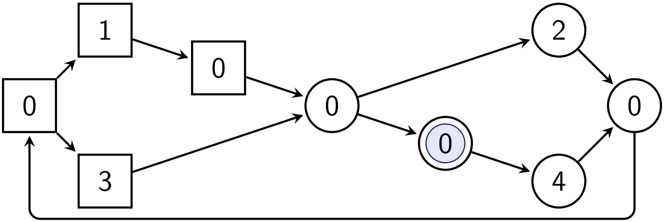


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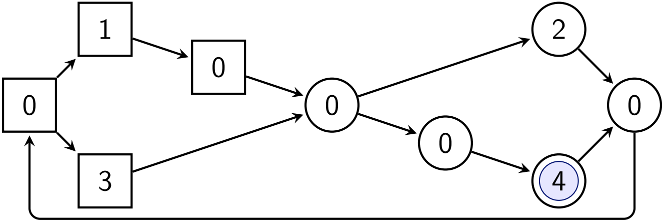


$0 \longrightarrow 1 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 4$

Example due to (Fijalkow and Chatterjee, Infinite-state games, 2013)

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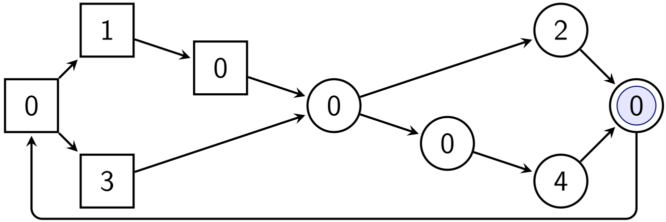


$$0 \longrightarrow 1 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 4 \longrightarrow 0$$

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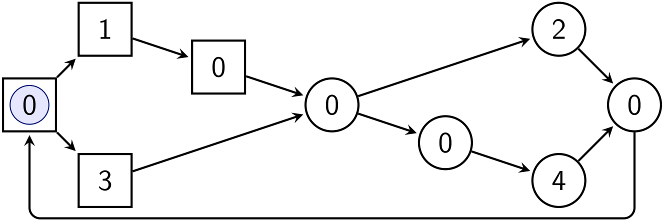


$0 \to 1 \to 0 \to 0 \to 0 \to 4 \to 0$

$\hookrightarrow 0$

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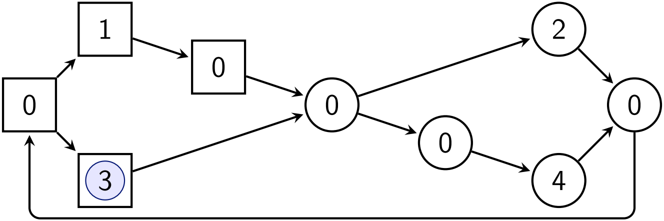


$0 \rightarrow 1 \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow 4 \rightarrow 0$

$0 \rightarrow 3$

Example due to (Fijalkow and Chatterjee, Infinite-state games, 2013)

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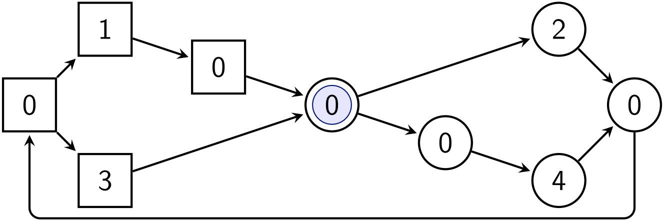


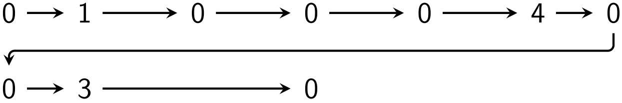


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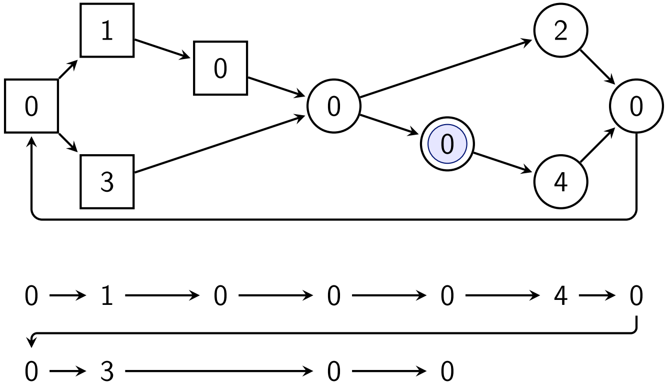


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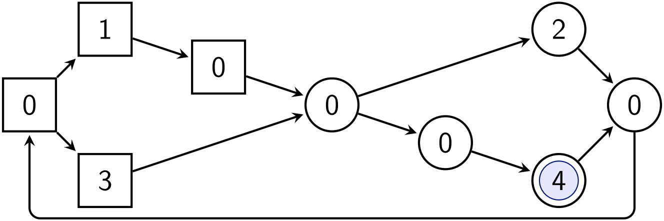


$$0 \rightarrow 1 \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow 4 \rightarrow 0$$

$$0 \rightarrow 3 \rightarrow 0 \rightarrow 0 \rightarrow 4$$

Example due to (Fijalkow and Chatterjee, Infinite-state games, 2013)

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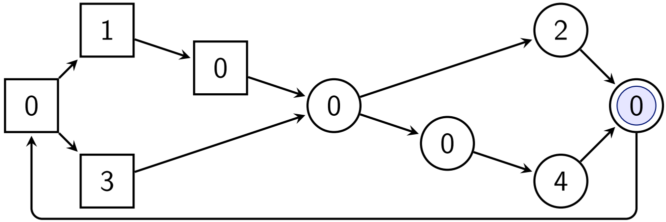


$$0 \rightarrow 1 \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow 4 \rightarrow 0$$

$$0 \rightarrow 3 \rightarrow 0 \rightarrow 0 \rightarrow 4 \rightarrow 0 \rightarrow \cdots$$

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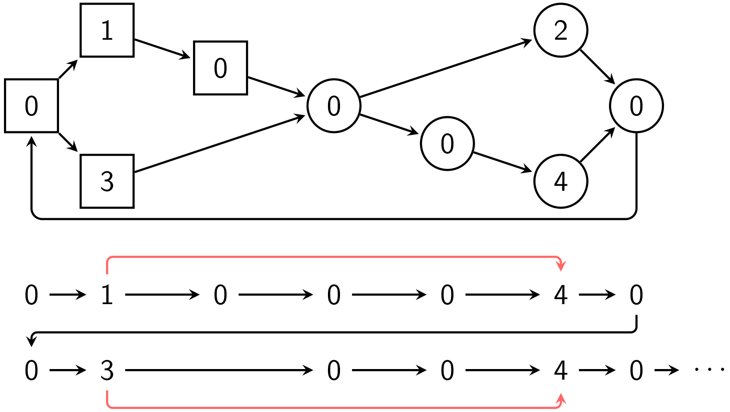


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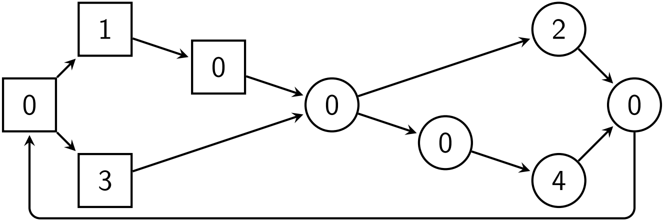
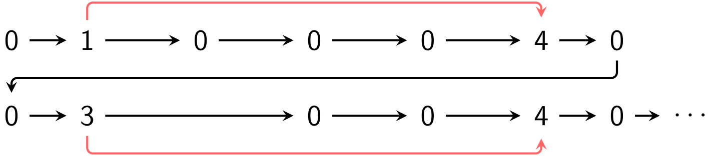


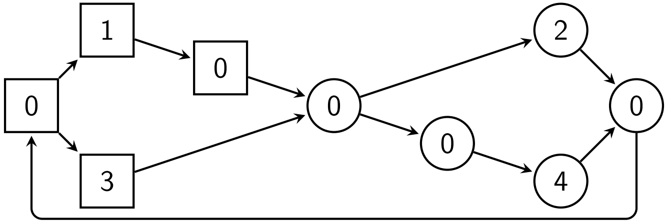




Deciding winner in $\mathrm{UP} \cap \mathrm{co}\text{-}\mathrm{UP}$

Positional Strategies

Example due to (Fijalkow and Chatterjee, Infinite-state games, 2013)

# Finitary Parity Games

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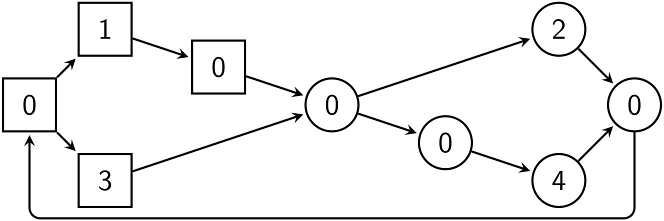


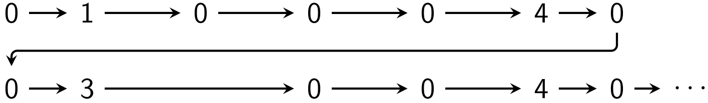


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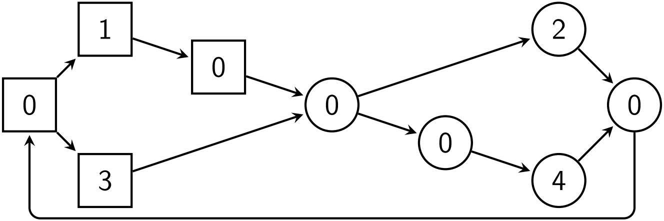


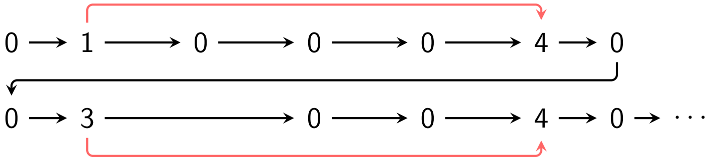


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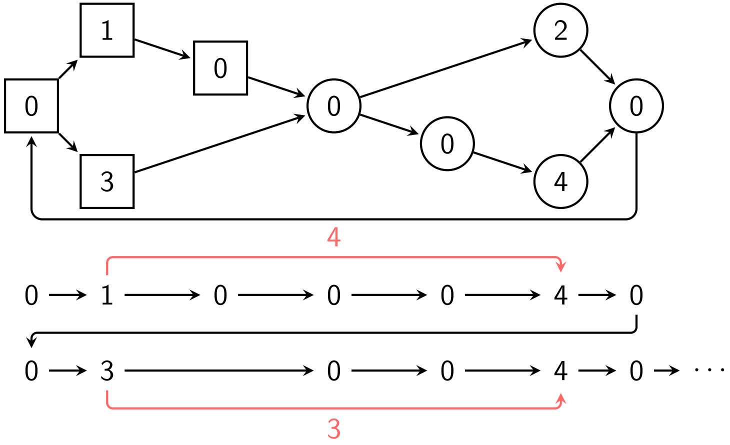


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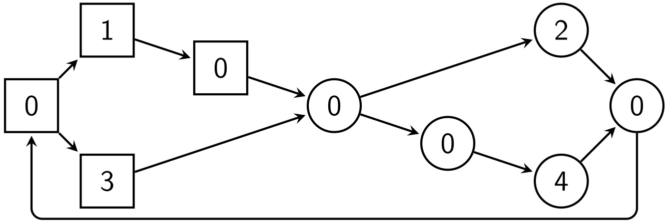


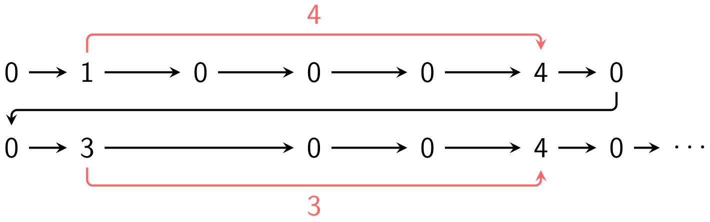


Goal for Player 0: Bound response times

Example due to (Fijalkow and Chatterjee, Infinite-state games, 2013)

# Decision Problem

## Theorem (Chatterjee et al., Finitary Winning, 2009)

*The following decision problem is in* $\mathrm{PT{\scriptstyle IME}}$*:*

**Input:** *Finitary parity game* $\mathcal{G} = (\mathcal{A}, \mathrm{FinParity}(\Omega))$

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*The following decision problem is* PSPACE*-complete:*

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# From Finitary Parity to Parity

**Given:** Finitary parity game $\mathcal{G} = (\mathcal{A}, \mathrm{FinParity}(\Omega))$, bound $b \in \mathbb{N}$.

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*Deciding if Player* $0$ *has strategy* $\sigma$ *with* $\mathrm{Cst}(\sigma) \leq b$ *is in* $\mathrm{PS{\small PACE}}$.

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(Chandra et al., Alternation, 1981)

$\Rightarrow$ Problem is in $\mathrm{PSPACE}$

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$\Rightarrow$ The given decision problem is PSPACE-complete

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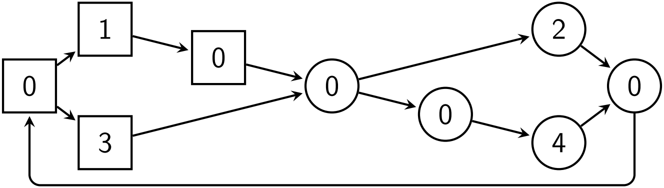

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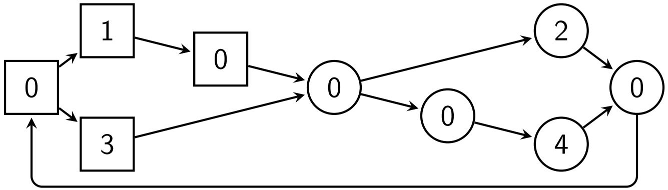


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- Generalize to $d$ colors
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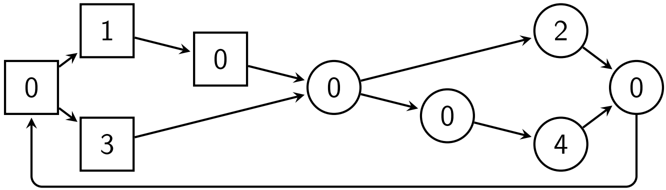


For given parameter $d$:

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- Repeat $d$ times

Player 1 has $d$ choices of $d$ actions

$\Rightarrow$

Player 0 needs $\approx 2^d$ memory states

# Conclusion

| | Parity | |
|---|---|---|
| Complexity | $\mathrm{UP} \cap \mathrm{co}\text{-}\mathrm{UP}$ | |
| Strategies | 1 | |

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**Take-away:** Forcing Player 0 to answer quickly in (finitary) parity games makes it harder

- to decide whether she can satisfy the bound
- for her to play the game