On Recurrent Reachability for Continuous Linear Dynamical Systems

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Exponential polynomials

Dot product $f(t) = \mathbf{v}^T \mathbf{x}(t)$, where

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Solns of IVP:

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'continuous linear recurrence sequences'

Let $\dot{\mathbf{x}}(t) = A\mathbf{x}(t)$. Intersections with plane?

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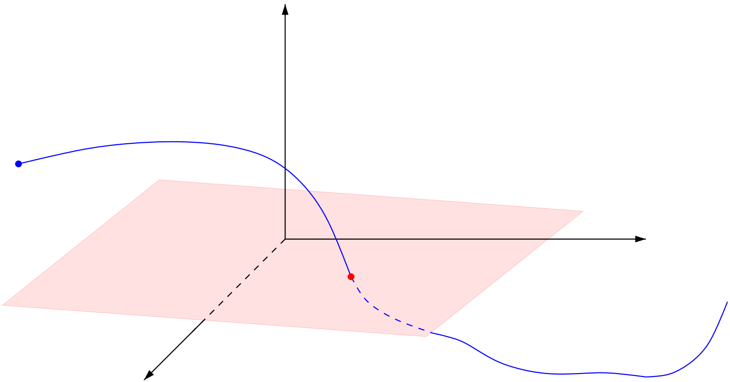

$\mathbf{x}$ intersects plane at time $t \iff \mathbf{v}^\mathsf{T}\mathbf{x}(t) = 0$

RecReach(k) problem:

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RecReach(9) hard for problems in Diophantine Approximation.

Continued Fractions

Finite continued fractions:

$$[3, 7, 15, 1, 292] = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292}}}}$$

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Infinite continued fractions:

$$[a_0, a_1, a_2, a_3, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

Diophantine approximation

Our hardness result

If RecReach(9) decidable, then the set

$$\{x \in \mathbb{R} \cap \mathbb{A} : x \text{ has bounded partial quotients}\}$$

is recursively enumerable.

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Oscillations: caused by complex eigenvalues.

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Hardness result

RecReach(9) decidable $\Rightarrow L_\infty(x)$ approximable for algebraic x.