A New Perspective on FO Model Checking of Dense Graph Classes

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Joint work with
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FO model checking

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Input: Structure S and a FO sentence φ

Question: Does $S \models \varphi$ hold?

- fundamental problem
- *algorithmic metatheorem*
- complexity:
  - PSPACE-complete Stockmeyer, Vardi
  - algorithm: $O(n^{f(\varphi)})$ (XP)

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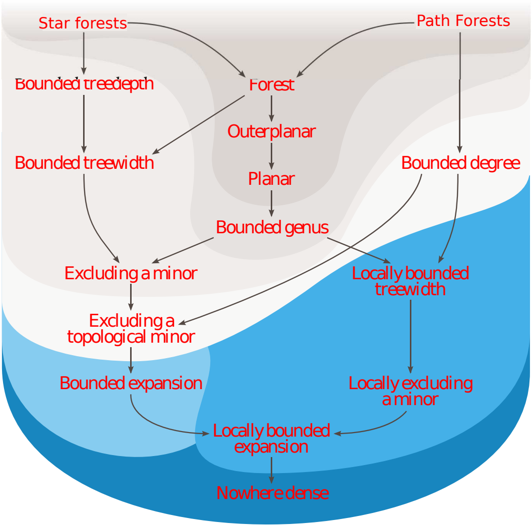
On which classes of structures can we do better?

Better: *fixed-parameter tractable* (FPT, i.e. $f(\varphi) \cdot n^{O(1)}$)

Sparse graph classes



FO on sparse graphs

FPT algorithms

- bounded degree graphs — Seese
- locally bounded treewidth — Frick, Grohe [ICALP'99]
- locally excluding a minor — Dawar, Grohe, Kreutzer [LICS'07]
- locally bounded expansion — Dvořák, Kráľ, Thomas [FOCS'10]
- nowhere dense graphs — Grohe, Kreutzer, Siebertz [STOC'14]

FO on dense graphs

Existing results

- L-interval graphs

  Ganian, Hliněný, Kráľ, O., Schwartz, Teska [ICALP'13]

- partially ordered sets

  Hliněný, Gajarský, Lokshtanov, O., Ordyniak, Saurabh, Ramanujan [FOCS'15]

Fact

- There is no nice structural theory of dense graphs which could be useful in this context.

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Fact

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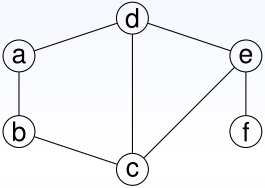
In this talk we:

- consider those *dense graphs* which can be *interpreted* in some sparse class of graphs
- identify a *subclass* for which we can solve the FO model checking problem in *FPT time*

(FO) Graph interpretations

$$\psi(x,y) \equiv x \neq y \wedge \exists z : (z \neq x) \wedge (z \neq y) \wedge edge(x,z) \wedge edge(z,y)$$



G

$$V(H) = V(G) \qquad E(H) = \{\{u,v\} \mid G \models \psi(u,v)\}$$

- H is *interpreted* in G using the formula $\psi(x,y)$
- generalized version: $V(H) \subseteq V(G)$
- naturally extends to classes of graphs

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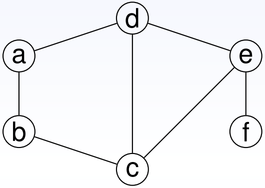

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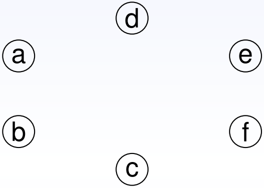

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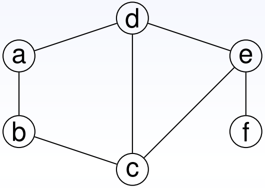

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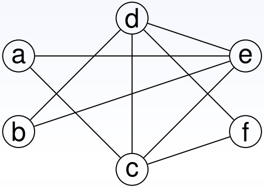

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Interpretations and model checking

Input: H, φ, an interpretation I_ψ and $G \in \mathcal{C}$ s.t. $H = I_\psi(G)$
Question: does $H \models \varphi$?

Assumption: for $\mathcal{C}$ there is an efficient model checking algorithm $\mathcal{A}$

The method

1. convert φ into φ' s.t. $H \models \varphi \iff G \models \varphi'$
2. solve $G \models \varphi'$ using $\mathcal{A}$

What if I_ψ and G are not given?

The questions we ask

Problem (1)

Let $\mathcal{D}$ be a graph class interpretable in a sparse graph class $\mathcal{C}$.
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Problem (2)
For a graph class $\mathcal{C}$, give a structural characterization of graph classes interpretable in $\mathcal{C}$.

Main result

Theorem

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To obtain this result we give a *full characterization* of graph classes interpretable in the class of graphs of bounded degree:

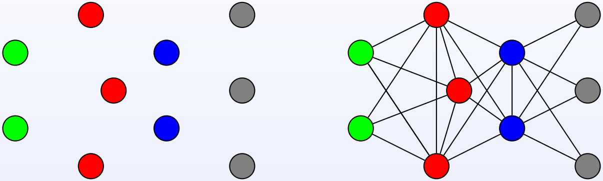
Theorem
Graph class $\mathcal{D}$ is interpretable in a class of graphs of bounded degree if, and only if, $\mathcal{D}$ is near-uniform.

Intermezzo: Neighbourhood diversity

- For a graph G, we say that two vertices $u, v \in V(G)$ are *twins* if $N(u) \setminus v = N(v) \setminus u$
- The twin relation is an *equivalence* relation on $V(G)$.
  (And each class is either a clique, or an independent set.)
- *Neighbourhood diversity* of a graph G is the number of equivalence classes of the twin relation.

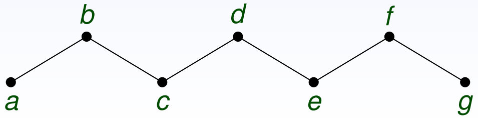
Interpretable in edgeless graphs

$$\psi(x,y) \equiv (Green(x) \wedge Red(y)) \vee (Red(x) \wedge Blue(y)) \vee (Blue(x) \wedge Gray(y)) \vee (Red(x) \wedge Red(y)) \vee (Blue(x) \wedge Blue(y))$$



Near-uniform graph classes

Definition
Two vertices $u, v \in V(G)$ are *near-k-twins* if $|N(u) \Delta N(v)| \leq k$.
We write $\{u, v\} \in \rho_k$



- ρ_1 classes: $\{a, c\}, \{e, g\}, \{b\}, \{d\}, \{f\}$ – an equivalence
- $(b, d) \in \rho_2$, $(d, f) \in \rho_2$, but $(b, f) \notin \rho_2$ – ρ_2 not an equivalence
- $\rho_4 = \binom{V}{2}$ – an equivalence

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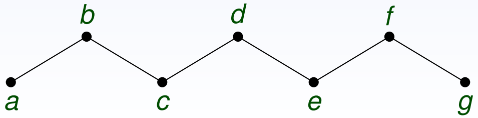

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Near-uniform graph classes

Definition
Graph class $\mathcal{C}$ is *near-uniform* if there exist k_0 and m such that for every $G \in \mathcal{C}$ there exists $k \leq k_0$ s.t. ρ_k is an equivalence on $V(G)$ with at most m classes.

Near-uniform graph classes 2

Definition
Graph class $\mathcal{C}$ is *near-uniform* if there exist k_0 and m such that for every $G \in \mathcal{C}$ there exists $k \leq k_0$ s.t. ρ_k is an equivalence on $V(G)$ with at most m classes.

Examples:

- graphs of degree at most d $(k_0 = 2d, m = 1)$
- complements of graphs of degree at most d $(k_0 = 2d, m = 1)$
- complete bipartite graphs $(k_0 = 0, m = 2)$
- complete bipartite graphs minus a matching $(k_0 = 2, m = 2)$

FO model checking algorithm

Input: $H \in \mathcal{D}$ (where $\mathcal{D}$ is (k_0, m)-near-uniform), FO formula φ
Output: YES iff $H \models \varphi$

Algorithm

1. For $k = 0, \ldots, k_0$ compute the near-k-twin relation ρ_k on $V(H)$ and check whether it is an equivalence.
   (Guaranteed to succeed for some $0 \leq k \leq k_0$)
2. Using ρ_k, compute a graph G of a bounded degree and a small formula $\psi(x, y)$ such that $H = I_\psi(G)$.
3. Compute φ' from φ (uses $\psi(x, y)$).
4. Use some efficient (known) model checking algorithm for graphs of bounded degree to determine whether $G \models \varphi'$.

Questions and open problems

Conjecture
Let $\mathcal{C}$ be a nowhere dense graph class and $\mathcal{D}$ a graph class interpretable in $\mathcal{C}$. Then $\mathcal{D}$ has an FO model checking algorithm in FPT.

Question
Can one characterize under which conditions on a formula $\psi(x, y)$ and a graph class $\mathcal{C}$, the following holds? Given a graph $H \in \mathcal{D}$ as an input, it would be possible to compute in polynomial (or in FPT with respect to ψ and $\mathcal{C}$) time a graph $G \in \mathcal{C}$ such that $H = \psi(G)$.

Question
What are the structural properties of graphs which are preserved under interpretations?