Primitive sets of nonnegative matrices and synchronizing automata

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1. Synchronizing automata

Let $\mathscr{A}$ be a complete deterministic finite automaton

$\mathscr{A}$ is synchronizing if there exists a word w and a state f such that for all states q we have $q \cdot w = f$

We need neither initial nor final states

Any word w with this property is a reset word or synchronizing word for $\mathscr{A}$

Related to and appear in:

1. Engineering and industrial automation
2. Synchronizing codes
3. Synchronizing groups
4. Combinatorics on words (non-complete sets of words)

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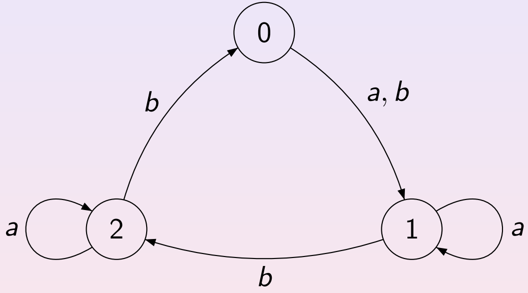
2. An example



The word *abba* is synchronizing

It resets the automaton to the state 1

The length of the shortest reset word is called reset threshold

How does the reset threshold depend on the number of states of automata?

Conjecture (Černý, 1964)

Every synchronizing automaton has a reset word of length at most $(n-1)^2$, where n is the number of states

Theorem (Pin, Frankl, 1983)

Every synchronizing automaton with n states has a reset word of length at most $\frac{n^3-n}{6}$

Confirmed in special cases: aperiodic, Eulerian, with a sink, with a cyclic letter, etc.

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4. Primitive matrices

A square and entrywise nonnegative matrix M is primitive if there exists t such that M^t is entrywise positive

Theorem (Perron, Frobenius, 1912)

Every primitive matrix has a unique largest real eigenvalue and the corresponding eigenvector can be chosen to have strictly positive components.

Related to and appear in:

1. Analysis of Markov chains
2. Economics (Okishio's theorem, Leontief's input-output model)
3. Demography (Leslie population age distribution model)
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5. Exponents and reset thresholds

The exponent $\exp(M)$ is the length of the shortest positive product

Theorem (Wielandt, 1950)

Exponent of every primitive $n \times n$ matrix does not exceed $(n-1)^2 + 1$ and the bound is tight

In 1980s all matrices with "large" exponent are described!

Key idea (Ananichev, G., Volkov, 2010)

Matrices with "large" exponent can be used to construct various series of synchronizing automata with the reset threshold close to $(n-1)^2$.

Automata theory benefited from matrix theory

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6. Primitive sets of matrices

Time to give back!

A finite set of square and entrywise nonnegative matrices $\mathcal{M} = \{M_1, M_2, \ldots, M_k\}$ is primitive if $M_{i_1} M_{i_2} \ldots M_{i_m}$ is entrywise positive for some indices $i_1, i_2, \ldots, i_m \in 1, \ldots, k$

Related to and appear in:

1. Stochastic control theory
2. Consensus problems
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7. Impact of automata theory

Key idea (Gerencsér, G., Jungers, 2016)

The study of primitive sets of matrices is "equivalent" to the study of synchronizing properties of (partial) automata.

$\exp(\mathcal{M})$ is the length of the shortest positive product

Theorem (Gerencsér, G., Jungers, 2016)

Let $\exp(n)$ *be the maximum value of the exponent among all sets of* $n \times n$ *matrices. We have* $\exp(n) \approx 3^{\frac{n}{3}}$.

Reduction from the Rystsov's result of 1980

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What about complete automata and standard synchronization?

A matrix set $\mathcal{M}$ is allowable if every matrix in $\mathcal{M}$ has no zero rows and columns

Importance:

1. has a nice generalization of Perron-Frobenius theory
2. is a tractable class of primitive sets of matrices
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9. Bounds on the exponents of allowable matrix sets

Let $\exp_{NZ}(n)$ be the largest exponent among *allowable* sets of matrices of size $n \times n$

We identify a special fixed class of complete automata $\mathcal{C}$ such that:

Theorem (Gerencsér, G., Jungers, 2016)

If $\text{syn}_{\mathcal{C}}(n)$ *is the largest reset threshold among n-state synchronizing automata from* $\mathcal{C}$, *then* $\exp_{NZ}(n) = \Theta(\text{syn}_{\mathcal{C}}(n))$.

Note that $\exp_{NZ}(n) = O(n^3)$

Open problem: prove that $\exp_{NZ}(n) = O(n^2)$ or demonstrate that such bound will be a big advancement on the Černý conjecture

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10. Complexity of problems for allowable matrix sets

A reduction from synchronizing automata to allowable matrix sets implies a strong theorem:

Theorem (Gerencsér, G., Jungers, 2016)

Given an allowable set $\mathcal{M}$ of three $n \times n$ matrices and possibly a positive integer k encoded in binary.

1. *The problem of deciding whether* $\exp(\mathcal{M}) \leq k$ *is NP-complete.*
2. *The problem of deciding whether* $\exp(\mathcal{M}) = k$ *is DP-complete.*
3. *The problem of computing* $\exp(\mathcal{M})$ *is* $FP^{NP[\log]}$*-complete.*
4. *For every constant* $\varepsilon > 0$ *it is NP-hard to approximate* $\exp(\mathcal{M})$ *within a factor* $n^{1-\varepsilon}$.

We utilize a large number of recent results from automata community

11. Conclusions

Main idea: there are strong connections between combinatorial matrix theory and automata theory

Properties of primitive sets of matrices are connected to synchronizing properties of partial automata

Properties of allowable primitive sets of matrices are connected to properties of a special subclass of deterministic automata

Open problems:

1. the Černý conjecture
2. upper bound on the exponent of an allowable matrix set