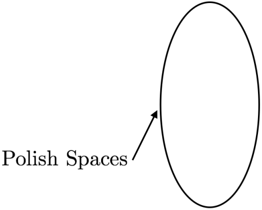
Towards a better Understanding of the Scott Domain

Louis Vuilleumier
joint work with Jacques Duparc

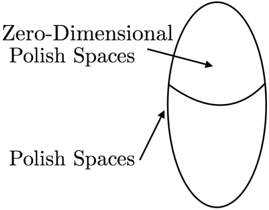
Department of Information Systems
University of Lausanne, Switzerland
Jacques.Duparc@unil.ch
Louis.Vuilleumier.1@unil.ch

September 7, 2016

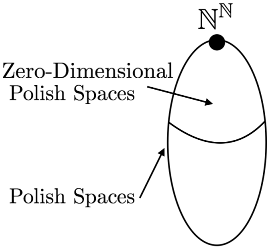
Classical Descriptive Set Theory



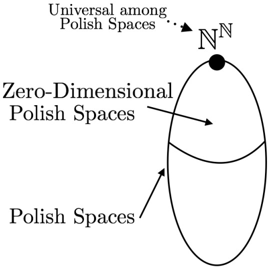
Classical Descriptive Set Theory



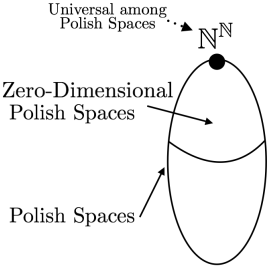
Classical Descriptive Set Theory



Classical Descriptive Set Theory

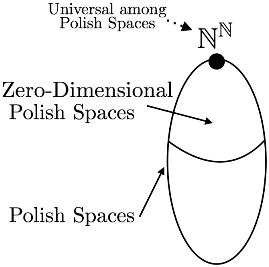


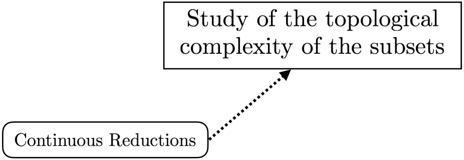
Classical Descriptive Set Theory



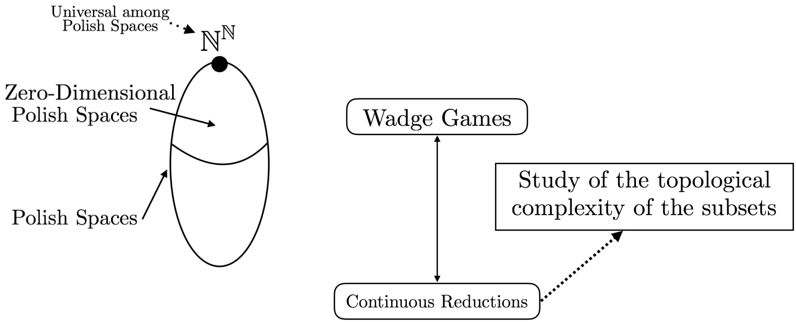
Study of the topological complexity of the subsets

Classical Descriptive Set Theory

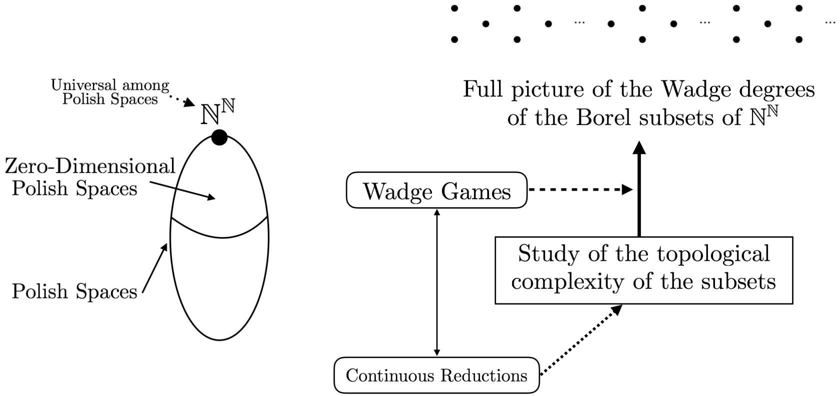




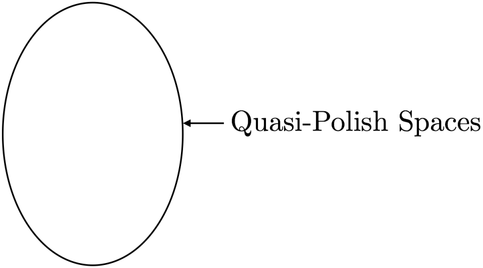
Classical Descriptive Set Theory



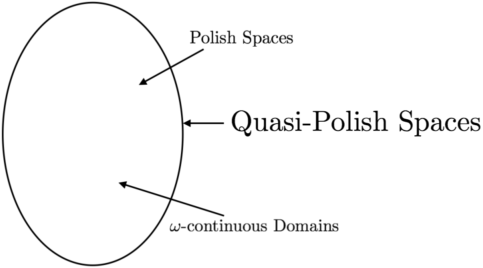
Classical Descriptive Set Theory



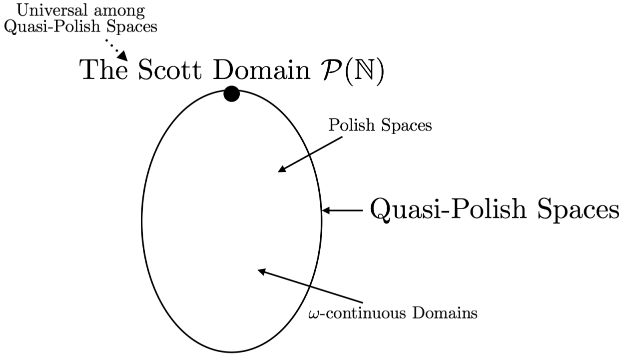
Generalization



Generalization

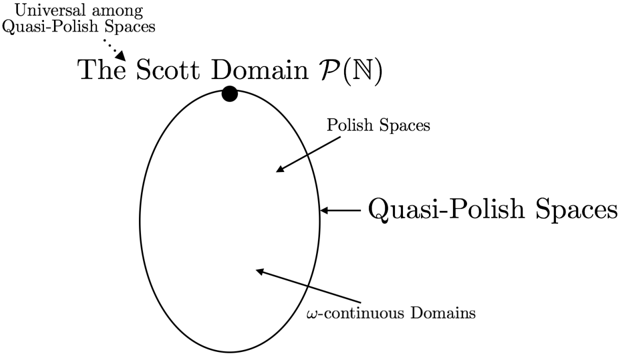


Generalization



Generalization

Games on $\mathcal{P}(\mathbb{N})$?



Scott domain and continuous reduction

Definition

Consider the set $\mathcal{P}(\mathbb{N})$ together with the topology — known as the **Scott topology** — generated by the basis

$$\mathcal{B} = \{\mathcal{O}_F : F \text{ finite subset of } \mathbb{N}\},$$

where

$$\mathcal{O}_F = \{X \subseteq \mathbb{N} : F \subseteq X\}.$$

This space is the **Scott domain**.

Scott domain and continuous reduction

Definition

Consider the set $\mathcal{P}(\mathbb{N})$ together with the topology — known as the **Scott topology** — generated by the basis

$$\mathcal{B} = \{\mathcal{O}_F : F \text{ finite subset of } \mathbb{N}\},$$

where

$$\mathcal{O}_F = \{X \subseteq \mathbb{N} : F \subseteq X\}.$$

This space is the **Scott domain**.

Theorem (de Brecht, 2013)

A space is quasi-Polish if and only if it is homeomorphic to some $\mathcal{A} \in \Pi^0_2(\mathcal{P}(\mathbb{N}))$.

Scott domain and continuous reduction

Definition

Consider the set $\mathcal{P}(\mathbb{N})$ together with the topology — known as the **Scott topology** — generated by the basis

$$\mathcal{B} = \{\mathcal{O}_F : F \text{ finite subset of } \mathbb{N}\},$$

where

$$\mathcal{O}_F = \{X \subseteq \mathbb{N} : F \subseteq X\}.$$

This space is the **Scott domain**.

Theorem (de Brecht, 2013)

A space is quasi-Polish if and only if it is homeomorphic to some $\mathcal{A} \in \Pi^0_2(\mathcal{P}(\mathbb{N}))$.

Definition

Let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(\mathbb{N})$. If there exists a continuous function $f : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ such that $f^{-1}(\mathcal{B}) = \mathcal{A}$, we say that $\mathcal{A}$ is **continuously reducible** or **Wadge reducible** to $\mathcal{B}$, and we write $\mathcal{A} \leq_W \mathcal{B}$.

Scott domain and continuous reduction

Definition

Consider the set $\mathcal{P}(\mathbb{N})$ together with the topology — known as the **Scott topology** — generated by the basis

$$\mathcal{B} = \{\mathcal{O}_F : F \text{ finite subset of } \mathbb{N}\},$$

where

$$\mathcal{O}_F = \{X \subseteq \mathbb{N} : F \subseteq X\}.$$

This space is the **Scott domain**.

Theorem (de Brecht, 2013)

A space is quasi-Polish if and only if it is homeomorphic to some $\mathcal{A} \in \Pi^0_2(\mathcal{P}(\mathbb{N}))$.

Definition

Let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(\mathbb{N})$. If there exists a continuous function $f : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ such that $f^{-1}(\mathcal{B}) = \mathcal{A}$, we say that $\mathcal{A}$ is **continuously reducible** or **Wadge reducible** to $\mathcal{B}$, and we write $\mathcal{A} \leq_W \mathcal{B}$.

$\leq_W$ induces a quasi-order relation on the subsets of the Scott domain.

Games on the Scott domain

Definition

Let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(\mathbb{N})$. We define a game $G_\infty(\mathcal{A}, \mathcal{B})$.

$$
\begin{array}{ccc}
I & & II \\
\hline
X_0 & \searrow & \\
\subseteq & & Y_0 \\
 & \swarrow & \\
X_1 & & \subseteq \\
 & \searrow & \\
\subseteq & & Y_1 \\
 & \swarrow & \\
X_2 & & \subseteq \\
 & \searrow & \\
\subseteq & & Y_2 \\
 & \swarrow & \\
\vdots & & \vdots \\
\hline
\end{array}
$$

$$X = \bigcup_{n \in \mathbb{N}} X_n, \quad Y = \bigcup_{n \in \mathbb{N}} Y_n$$

$$X_n \in \mathcal{P}_{<\infty}(\mathbb{N}), \ Y_n \in \mathcal{P}(\mathbb{N})$$

II wins if and only if
$(X \in \mathcal{A} \leftrightarrow Y \in \mathcal{B})$.

Strategy for *II*

Definition

An **ultrapositional strategy** for *II* in a game $G_\infty(\mathcal{A},\mathcal{B})$ is an increasing function $\sigma : \mathcal{P}_{<\infty}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$, i.e. such that for all $X_0, X_1 \in \mathcal{P}_{<\infty}(\mathbb{N})$ with $X_0 \subseteq X_1$, we have $\sigma(X_0) \subseteq \sigma(X_1)$.

Strategy for *II*

Definition

An **ultrapositional strategy** for *II* in a game $G_\infty(\mathcal{A}, \mathcal{B})$ is an increasing function $\sigma : \mathcal{P}_{<\infty}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$, i.e. such that for all $X_0, X_1 \in \mathcal{P}_{<\infty}(\mathbb{N})$ with $X_0 \subseteq X_1$, we have $\sigma(X_0) \subseteq \sigma(X_1)$.

Definition

An ultrapositional strategy is **winning** for *II* if, following this ultrapositional strategy and whatever *I* plays, *II* wins the game $G_\infty(\mathcal{A}, \mathcal{B})$.

Strategy for *II*

Definition

An **ultrapositional strategy** for *II* in a game $G_\infty(\mathcal{A}, \mathcal{B})$ is an increasing function $\sigma : \mathcal{P}_{<\infty}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$, i.e. such that for all $X_0, X_1 \in \mathcal{P}_{<\infty}(\mathbb{N})$ with $X_0 \subseteq X_1$, we have $\sigma(X_0) \subseteq \sigma(X_1)$.

Definition

An ultrapositional strategy is **winning** for *II* if, following this ultrapositional strategy and whatever *I* plays, *II* wins the game $G_\infty(\mathcal{A}, \mathcal{B})$.

Proposition

Let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(\mathbb{N})$. The following are equivalent.

1. $\mathcal{A} \leq_W \mathcal{B}$,
2. *II* has a winning ultrapositional strategy in $G_\infty(\mathcal{A}, \mathcal{B})$.

Examples

Consider $G_\infty\left(\{\mathbb{N}\}, \mathcal{P}_\infty(\mathbb{N})\right)$.

$$\begin{aligned}\sigma : \mathcal{P}_{<\infty}(\mathbb{N}) &\to \mathcal{P}(\mathbb{N}) \\ X &\mapsto \bigcup_{\substack{n \in \mathbb{N} \\ \{0,\ldots,n\} \subseteq X}} \{0, \ldots, n\}.\end{aligned}$$

It is a ultrapositional winning strategy.

Hence, $\{\mathbb{N}\} \leq_W \mathcal{P}_\infty(\mathbb{N})$.

Examples

Consider $G_\infty(\{\mathbb{N}\}, \mathcal{P}_\infty(\mathbb{N}))$.

$$\sigma : \mathcal{P}_{<\infty}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$$
$$X \mapsto \bigcup_{\substack{n\in\mathbb{N} \\ \{0,\dots,n\}\subseteq X}} \{0,\dots,n\}.$$

It is a ultrapositional winning strategy.

Hence, $\{\mathbb{N}\} \leq_W \mathcal{P}_\infty(\mathbb{N})$.

Consider $G_\infty(\{\{0\}\}, \{\{0\},\{0,1,2\}\})$.

$$\sigma : \mathcal{P}_{<\infty}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$$
$$\emptyset \mapsto \emptyset,$$
$$\{0\} \mapsto \{0\},$$
$$X \mapsto \{0,1\} \quad \text{otherwise.}$$

It is a ultrapositional winning strategy.

Hence, $\{\{0\}\} \leq_W \{\{0\},\{0,1,2\}\}$.

Examples

Consider the game $G_\infty(\{\mathbb{N}\}, \{\{0\}\})$.

I	*II*
$\varnothing$	

Examples

Consider the game $G_\infty(\{\mathbb{N}\}, \{\{0\}\})$.

I	*II*
$\varnothing$	
	$\varnothing$

Examples

Consider the game $G_\infty(\{\mathbb{N}\}, \{\{0\}\})$.

I	*II*
$\varnothing$	
	$\varnothing$
$\{0\}$	

Examples

Consider the game $G_\infty(\{\mathbb{N}\}, \{\{0\}\})$.

I	*II*
$\varnothing$	
	$\varnothing$
$\{0\}$	
	$\varnothing$

Examples

Consider the game $G_\infty(\{\mathbb{N}\}, \{\{0\}\})$.

I	*II*
$\varnothing$	
	$\varnothing$
$\{0\}$	
	$\varnothing$
$\{0, 1\}$	

Examples

Consider the game $G_\infty(\{\mathbb{N}\}, \{\{0\}\})$.

I	II
$\emptyset$	
	$\emptyset$
$\{0\}$	
	$\emptyset$
$\{0, 1\}$	
	$\emptyset$

Examples

Consider the game $G_\infty(\{\mathbb{N}\}, \{\{0\}\})$.

I	II
$\emptyset$	
	$\emptyset$
$\{0\}$	
	$\emptyset$
$\{0, 1\}$	
	$\emptyset$
$\{0, 1, 2\}$	

Examples

Consider the game $G_\infty(\{\mathbb{N}\}, \{\{0\}\})$.

I	II
$\varnothing$	
	$\varnothing$
$\{0\}$	
	$\varnothing$
$\{0,1\}$	
	$\varnothing$
$\{0,1,2\}$	
	$\{0\}$

Examples

Consider the game $G_\infty(\{\mathbb{N}\}, \{\{0\}\})$.

I	II
$\varnothing$	
	$\varnothing$
$\{0\}$	
	$\varnothing$
$\{0,1\}$	
	$\varnothing$
$\{0,1,2\}$	
	$\{0\}$
$\{0,1,2\}$	

Examples

Consider the game $G_\infty(\{\mathbb{N}\}, \{\{0\}\})$.

I	*II*
$\varnothing$	
	$\varnothing$
$\{0\}$	
	$\varnothing$
$\{0, 1\}$	
	$\varnothing$
$\{0, 1, 2\}$	
	$\{0\}$
$\{0, 1, 2\}$	
	$\{0\}$

Examples

Consider the game $G_\infty(\{\mathbb{N}\}, \{\{0\}\})$.

I	II
$\varnothing$	
	$\varnothing$
$\{0\}$	
	$\varnothing$
$\{0,1\}$	
	$\varnothing$
$\{0,1,2\}$	
	$\{0\}$
$\{0,1,2\}$	
	$\{0\}$
$\{0,1,2\}$	
	$\{0\}$

$\lightning\lightning\lightning$

Hence, $\{\mathbb{N}\} \not\leq_W \{\{0\}\}$.

Examples

Consider the game $G_\infty(\{\mathbb{N}\},\{\{0\}\})$.

I	II
$\varnothing$	
	$\varnothing$
$\{0\}$	
	$\varnothing$
$\{0,1\}$	
	$\varnothing$
$\{0,1,2\}$	
	$\{0\}$
$\{0,1,2\}$	
	$\{0\}$
$\{0,1,2\}$	
	$\{0\}$

$\lightning\lightning\lightning$

Hence, $\{\mathbb{N}\} \not\leq_W \{\{0\}\}$.

We also have :

$\left(\{\mathbb{N}\},\{\{0\}\},\{\mathbb{N}\}^C,\{\{0\}\}^C\right)$ is an antichain with respect to the quasi-order $\leq_W$ in the Scott domain.

No self-dual set in the Scott domain

Proposition

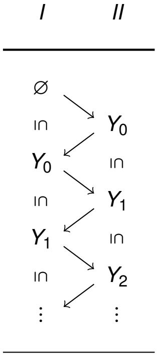
Let $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$, then $\mathcal{A}$ is non-self-dual.

No self-dual set in the Scott domain

Proposition

Let $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$, then $\mathcal{A}$ is non-self-dual.

Proof.



$$X = \bigcup_{n\in\mathbb{N}} Y_n, \qquad Y = \bigcup_{n\in\mathbb{N}} Y_n$$

□

Antichains in the Scott domain

Proposition (V.)

For every $k \in \mathbb{N} \setminus \{0\}$, there exists a sequence $(\mathcal{A}_1, \ldots, \mathcal{A}_k)$ such that, for all $i, j \in \{1, \ldots, k\},\ i \neq j$, we have $\mathcal{A}_i \subseteq \Delta^0_3(\mathcal{P}(\mathbb{N}))$ and $\mathcal{A}_i$ is not continuously reducible to $\mathcal{A}_j$.

Antichains in the Scott domain

Proposition (V.)

For every $k \in \mathbb{N} \setminus \{0\}$, there exists a sequence $(\mathcal{A}_1, \ldots, \mathcal{A}_k)$ such that, for all $i, j \in \{1, \ldots, k\},\ i \neq j$, we have $\mathcal{A}_i \subseteq \Delta^0_3(\mathcal{P}(\mathbb{N}))$ and $\mathcal{A}_i$ is not continuously reducible to $\mathcal{A}_j$.

Merci !