

The MSO+U Theory of $(\mathbb{N}, <)$ Is Undecidable

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MSO+U logic

MSO+U extends MSO by the following „U” quantifier:

$$\mathbf{UX}.\phi(X)$$

$\phi(X)$ holds for arbitrarily large finite sets

This construction may be nested inside other quantifiers,
and ϕ may have free variables other than X .

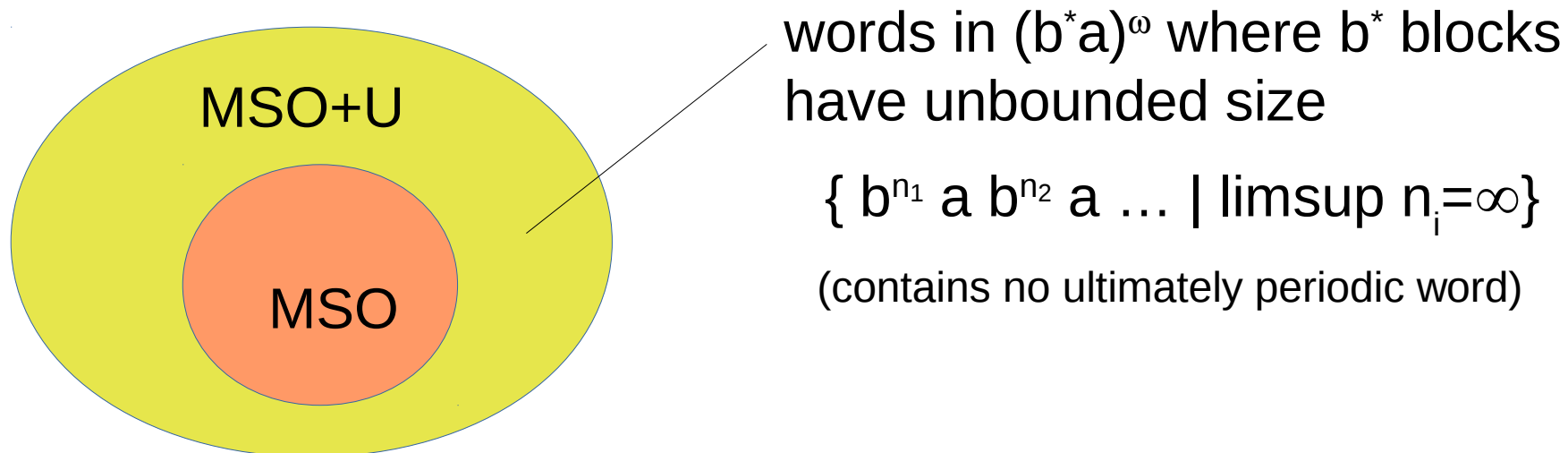
MSO+U was introduced by Bojańczyk in 2004

MSO+U logic – separating example

MSO+U extends MSO by the following „U” quantifier:

$$\mathbf{UX}.\phi(X)$$

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MSO+U logic is close to MSO

Fix ϕ and consider the following “Myhill-Nerode” relation:

$$v \sim v' \quad \text{when} \quad \text{for all words } u \in A^*, w \in A^\omega \quad (uvw \models \phi \Leftrightarrow uv'w \models \phi)$$

This relation has finitely many equivalence classes.

Slogan: The non-regularity of MSO+U is seen only in the asymptotic behavior.

Considered problem

satisfiability for ω -words

Input: formula $\phi \in \text{MSO+U}$

Question: does $w \models \phi$ for some $w \in A^\omega$?

Equivalently:

evaluation over $(\mathbb{N}, <)$

Input: formula $\phi \in \text{MSO+U}$

Question: $a^\omega \models \phi$?

Our result: Satisfiability of MSO+U for ω -words is undecidable!

MSO+U logic

Plan of the talk:

- 1) Some fragments of MSO+U are decidable – *earlier work*
 - a) BS-formulas
 - b) WMSO+U
- 2) MSO+U is not decidable over ω -words – ***this work***

Decidable fragments of MSO+U

negation allowed

BS-formulas: boolean combinations of
formulas in which U appears positively
(+ existential quantification outside)

Theorem (Bojańczyk & Colcombet, 2006):
Satisfiability of BS-formulas is decidable over ω -words.

Proof: Uses ω BS-automata.

These are nondeterministic automata with counters that can be incremented and reset to 0, but cannot be read. Accepting condition: counter is bounded/unbounded.

Decidable fragments of MSO+U

Weak logics: \exists/\forall quantifiers range only over finite sets.

Satisfiability is decidable for:

WMSO+U on infinite words (Bojańczyk, 2009)

WMSO+R on infinite words (Bojańczyk & Toruńczyk, 2009)

R = exists infinitely many sets of bounded size

WMSO+U on infinite trees (Bojańczyk & Toruńczyk, 2012)

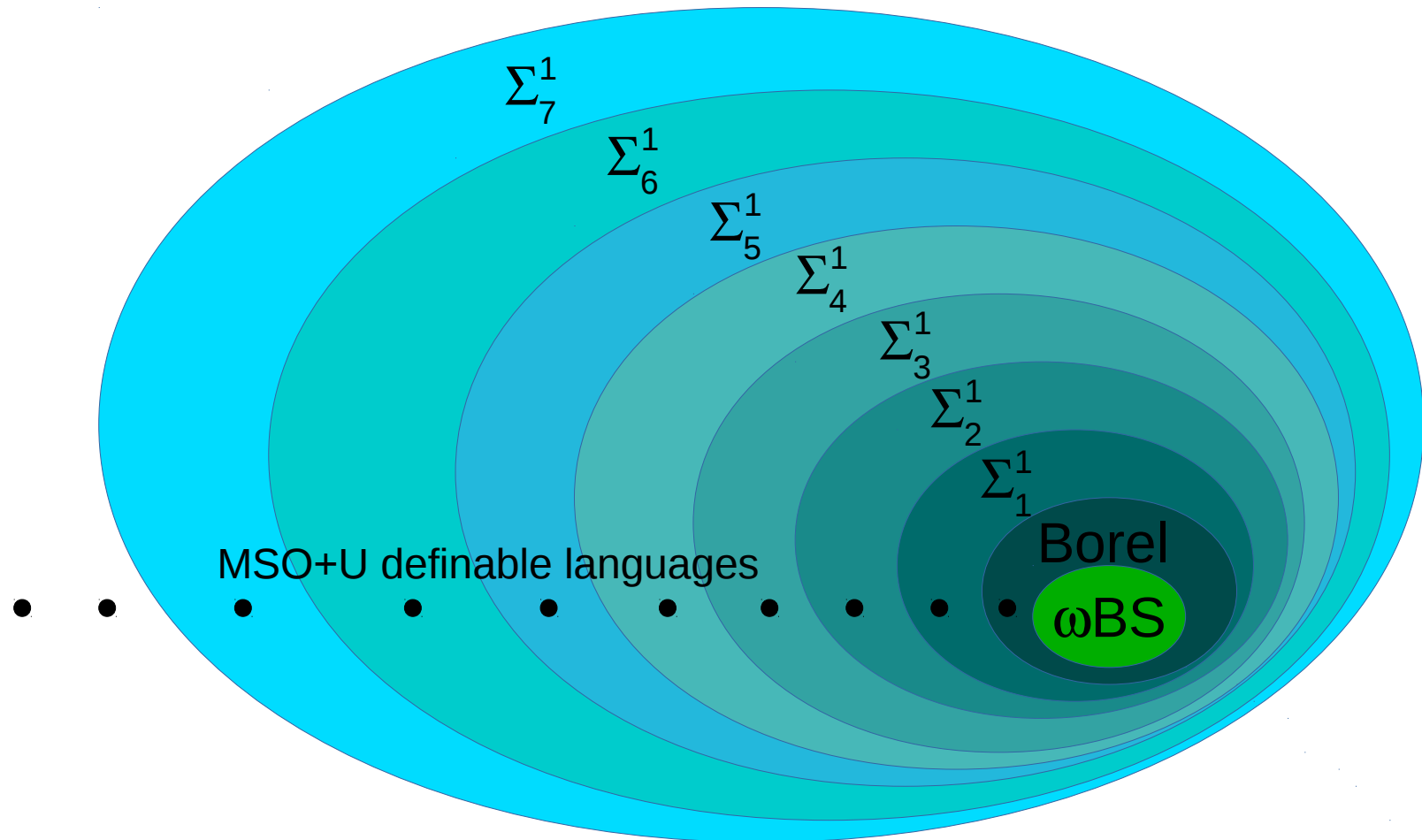
WMSO+U+P on infinite trees (Bojańczyk, 2014)

P = exists path

Proof: equivalent automata models

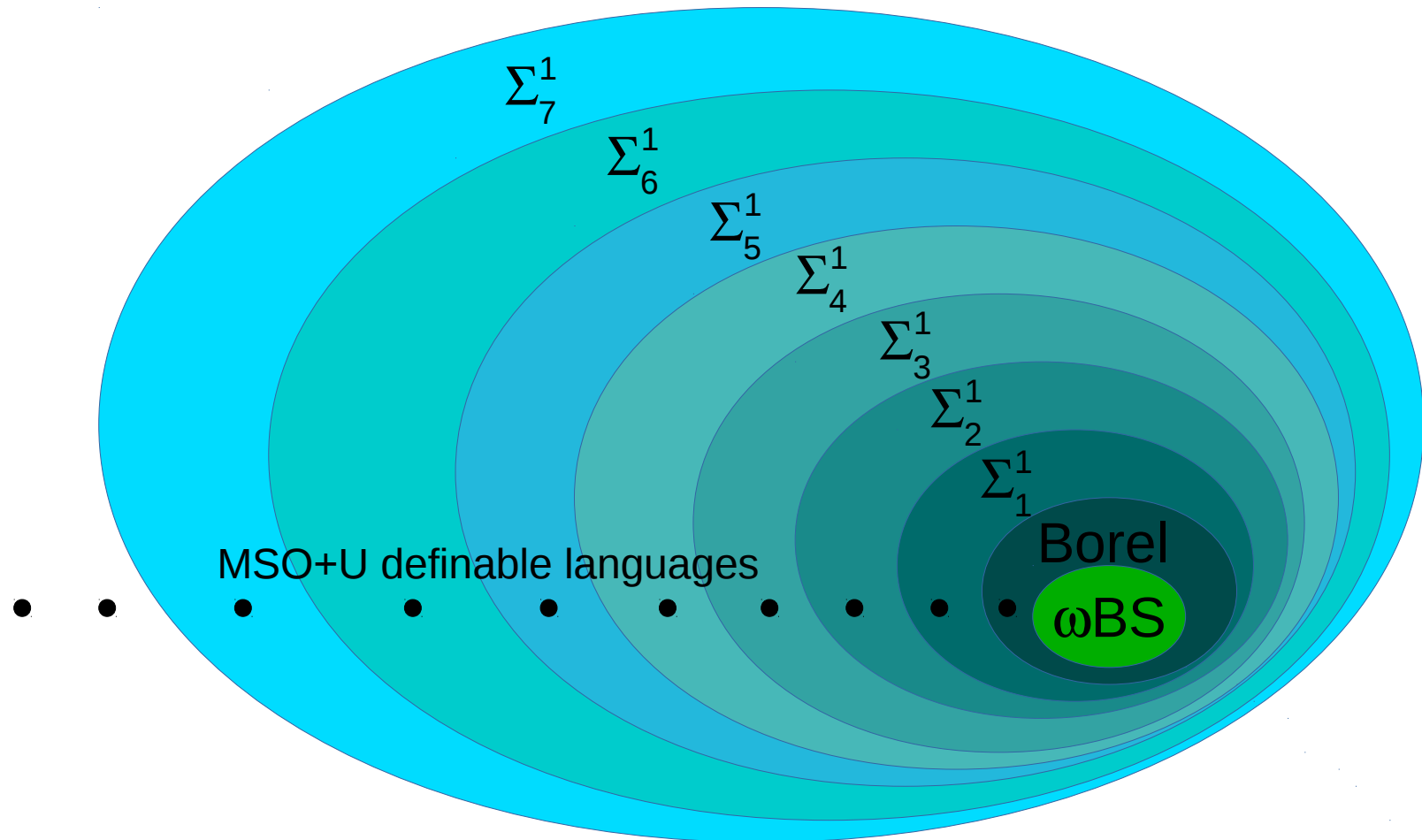
Undecidability of MSO+U – earlier work

Theorem. (Hummel & Skrzypczak 2010/2012) - topology
On every level of the projective hierarchy for infinite words, there is a complete language that is definable in MSO+U.



Undecidability of MSO+U – earlier work

Theorem. (Hummel & Skrzypczak 2010/2012) - topology
On every level of the projective hierarchy for infinite words, there is a complete language that is definable in MSO+U.



Corollary: MSO+U is not covered by any automata model
(alternating/nondeterm./determ., acceptance condition of bounded projective complexity)

Undecidability of MSO+U – earlier work

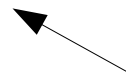
Theorem 1. (Hummel & Skrzypczak 2010/2012)

On every level of the projective hierarchy for infinite words, there is a complete language that is definable in MSO+U.

Theorem 2. (Bojańczyk, Gogacz, Michalewski, Skrzypczak 2014)

Satisfiability of MSO+U is not decidable over infinite trees...

...assuming that there exists a projective ordering on the Cantor set 2^ω .



assumption of set theory consistent with ZFC

Corollary: No algorithm can decide satisfiability of MSO+U over infinite trees and have a correctness proof in ZFC.

Proof:

Bases on Theorem 1 & the proof of Shelah that MSO is undecidable in 2^ω .

Altogether rather complicated.

Undecidability of MSO+U

Thm. (Bojańczyk, P., Toruńczyk 2016)

MSO+U is not decidable over infinite words.

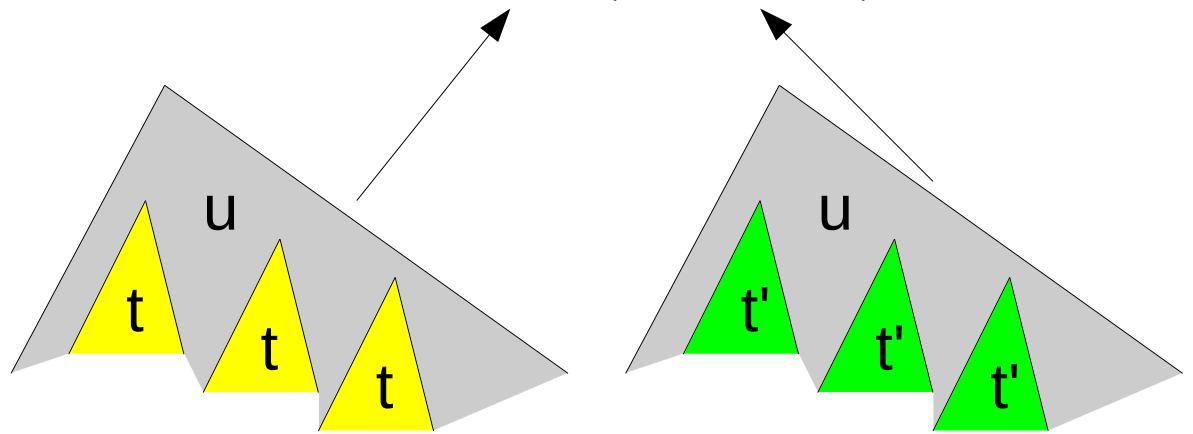
Proof – short & elementary (but too long for 10 minutes).

Details: „The MSO+U Theory of $(\mathbb{N}, <)$ Is Undecidable” at STACS 2016

Undecidability of MSO+U – core observation

Consider the following “Myhill-Nerode” relation for trees:

$t \sim t'$ when for all contexts u ($ut \models \phi \Leftrightarrow ut' \models \phi$)



For some ϕ , this relation has **infinitely** many equivalence classes.

Thank you!