A Dual to a Classical Graph Homomorphism Theorem of Lovász

Stefan Mengel

CNRS, CRIL UMR 8188
(joint work with Hubie Chen)

08/09/2016

Graph Homomorphisms

Definition (Graph Homomorphism)

Homomorphism between G and H: Mapping $h : V(G) \to V(H)$ that maps edges onto edges.

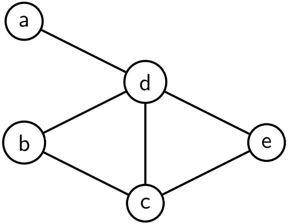
all graphs undirected, finite

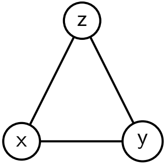
Graph Homomorphisms

Definition (Graph Homomorphism)

Homomorphism between G and H: Mapping $h : V(G) \to V(H)$ that maps edges onto edges.

all graphs undirected, finite



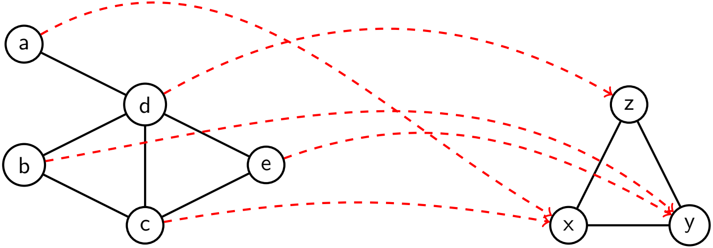


Graph Homomorphisms

Definition (Graph Homomorphism)

Homomorphism between G and H: Mapping $h : V(G) \to V(H)$ that maps edges onto edges.

all graphs undirected, finite, allow self-loops

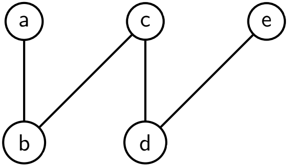


Graph Homomorphisms

Definition (Graph Homomorphism)

Homomorphism between G and H: Mapping $h : V(G) \to V(H)$ that maps edges onto edges.

all graphs undirected, finite

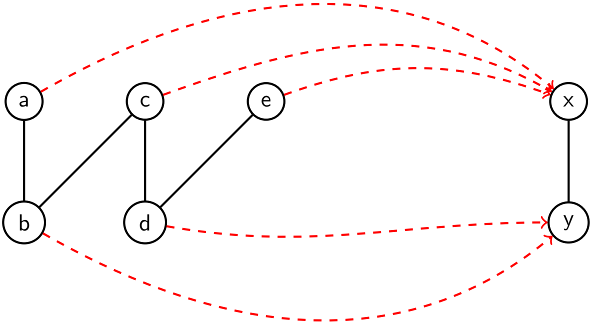


Graph Homomorphisms

Definition (Graph Homomorphism)

Homomorphism between G and H: Mapping $h : V(G) \to V(H)$ that maps edges onto edges.

all graphs undirected, finite

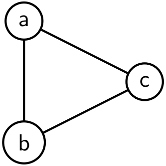


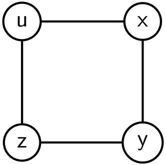
Graph Homomorphisms

Definition (Graph Homomorphism)

Homomorphism between G and H: Mapping $h : V(G) \rightarrow V(H)$ that maps edges onto edges.

all graphs undirected, finite





Some Background

- graph homomorphisms (and generalizations) studied in many fields
  - structural graph theory
  - artificial intelligence: constraint satisfaction
  - databases: conjunctive queries
  - statistical physics
- homomorphism perspective often very "clean", allows algebraic perspective

The Complexity of Graph Homomorphisms

Theorem

It is NP*-hard to decide, given two graphs* G, H*, if there is a homomorphism from* G *to* H*.*

The Complexity of Graph Homomorphisms

Theorem

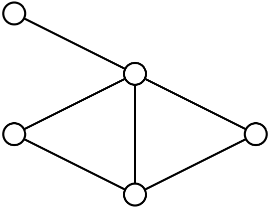
It is NP*-hard to decide, given two graphs* G, H*, if there is a homomorphism from* G *to* H*.*

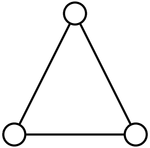
Lemma

For every graph G *the following is equivalent:*

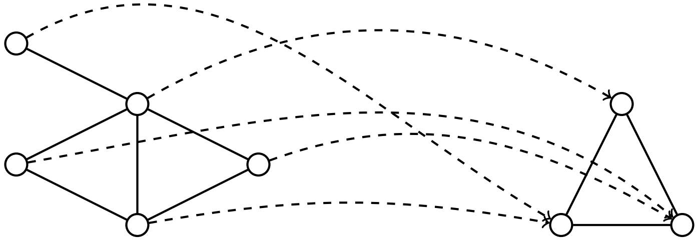
- G *has a homomorphism to the triangle graph*
- G *is* 3*-colorable*

Homomorphisms to the Triangle

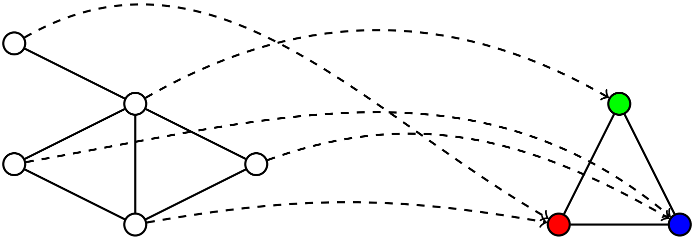




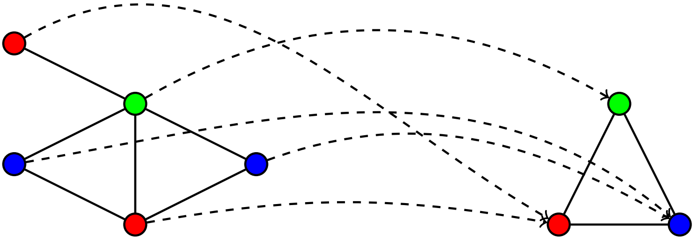
Homomorphisms to the Triangle



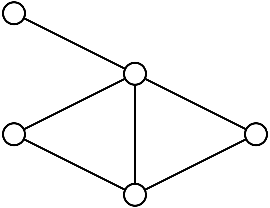
Homomorphisms to the Triangle

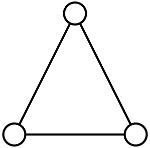


Homomorphisms to the Triangle

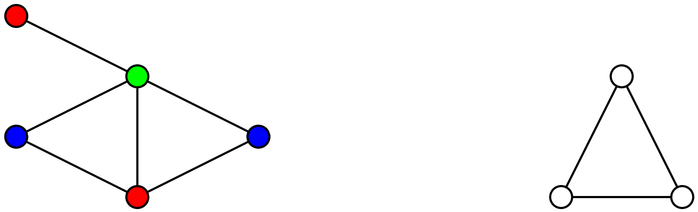


Homomorphisms to the Triangle

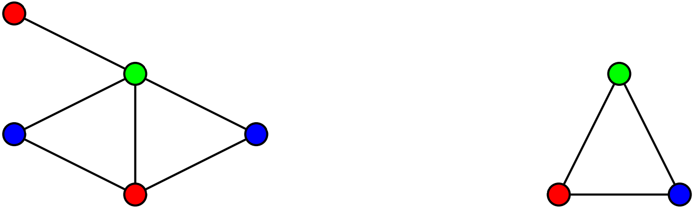




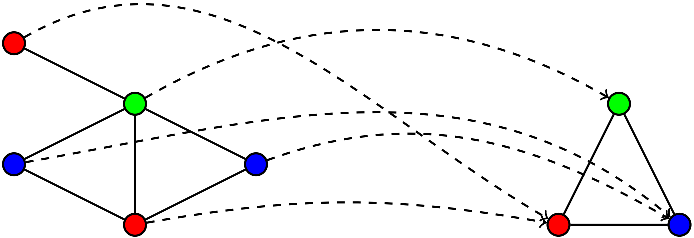
Homomorphisms to the Triangle



Homomorphisms to the Triangle



Homomorphisms to the Triangle



The Complexity of Graph Homomorphisms

Theorem

Given two graphs, decide if they are isomorphic.

Lemma

For every graph G the following is equivalent:

- *G has a homomorphism to the triangle graph*
- *G is 3-colorable*

- easily generalizes to k-colorability for every k

Counting Homomorphisms

- so homomorphisms carry information about G. how much?
- $\hom(G, H)$: set of homomorphisms from G to H

Counting Homomorphisms

- so homomorphisms carry information about G. how much?
- $\hom(G, H)$: set of homomorphisms from G to H

Theorem (Lovász 1967)

Graphs H_1, H_2 are isomorphic, if and only if for every graph G

$$|\hom(G, H_1)| = |\hom(G, H_2)|$$

Counting Homomorphisms

- so homomorphisms carry information about G. how much?
- $\hom(G, H)$: set of homomorphisms from G to H

Theorem (Lovász 1967)

Graphs H_1, H_2 are isomorphic, if and only if for every graph G

$$|\hom(G, H_1)| = |\hom(G, H_2)|$$

- so number of homomorphisms to H contains all information on H (up to isomorphism)

Counting Homomorphisms

- so homomorphisms carry information about G. how much?
- $\hom(G, H)$: set of homomorphisms from G to H

Theorem (Lovász 1967)

Graphs H_1, H_2 are isomorphic, if and only if for every graph G

$$|\hom(G, H_1)| = |\hom(G, H_2)|$$

- so number of homomorphisms to H contains all information on H (up to isomorphism)
- optimal
  - just existence (and not count) is not enough $\rightsquigarrow$ homomorphic equivalence
  - isomorphic graphs have same number of homomorphisms

Counting From the Other Size

Theorem

Graphs G_1, G_2 are isomorphic, if and only if for every graph H

$$|\operatorname{hom}(G_1, H)| = |\operatorname{hom}(G_2, H)|$$

Counting From the Other Size

Theorem

Graphs G_1, G_2 are isomorphic, if and only if for every graph H

$$|\hom(G_1, H)| = |\hom(G_2, H)|$$

Theorem (Equivalent formulation)

For all non-isomorphic G_1, G_2, there is H with

$$|\hom(G_1, H)| \neq |\hom(G_2, H)|.$$

Counting From the Other Size

Theorem

Graphs G_1, G_2 are isomorphic, if and only if for every graph H

$$|\hom(G_1, H)| = |\hom(G_2, H)|$$

Theorem (Equivalent formulation)

For all non-isomorphic G_1, G_2, there is H with

$$|\hom(G_1, H)| \neq |\hom(G_2, H)|.$$

- number of homomorphism from G describes G up to isomorphism
- natural dual version of Lovász' result
- optimality in the same sense as before

Proof Ideas

Theorem (Equivalent formulation)

For all non-isomorphic G_1, G_2, there is H with

$$|\hom(G_1, H)| \neq |\hom(G_2, H)|.$$

Proof Ideas

Theorem (Equivalent formulation)

For all non-isomorphic G_1, G_2, there is H with

$$|\hom(G_1, H)| \neq |\hom(G_2, H)|.$$

- G_1, G_2 different size easy to handle, so same size

Proof Ideas

Theorem (Equivalent formulation)

For all non-isomorphic G_1, G_2, there is H with

$$|\hom(G_1, H)| \neq |\hom(G_2, H)|.$$

- G_1, G_2 different size easy to handle, so same size
- no surjection between G_1 and G_2

Proof Ideas

Theorem (Equivalent formulation)

For all non-isomorphic G_1, G_2, there is H with

$$|\operatorname{hom}(G_1, H)| \neq |\operatorname{hom}(G_2, H)|.$$

- G_1, G_2 different size easy to handle, so same size
- no surjection between G_1 and G_2
- inclusion-exclusion

$$|\operatorname{surj}(G_1, G_2)| = \sum_{T \subseteq V(G_2)} (-1)^{|V(G_2)| - |T|} |\operatorname{hom}(G_1, G_2[T])|,$$

where $G_2[T]$ subgraph of G_2 induced by T

Proof Ideas

Theorem (Equivalent formulation)

For all non-isomorphic G_1, G_2, there is H with

$$|\operatorname{hom}(G_1, H)| \neq |\operatorname{hom}(G_2, H)|.$$

- G_1, G_2 different size easy to handle, so same size
- no surjection between G_1 and G_2
- inclusion-exclusion

$$|\operatorname{surj}(G_1, G_2)| = \sum_{T \subseteq V(G_2)} (-1)^{|V(G_2)| - |T|} |\operatorname{hom}(G_1, G_2[T])|,$$

  where $G_2[T]$ subgraph of G_2 induced by T
- compare terms: $0 = |\operatorname{surj}(G_1, G_2)| = |\operatorname{surj}(G_2, G_2)| \neq 0$

Conclusion

- gave dual version to classical theorem by Lovász

Conclusion

- gave dual version to classical theorem by Lovász
- generalizations
  - works for every finite set of nonisomorphic graphs
  - "primitive positive formulas" / "conjunctive queries", arbitrary finite structures instead of graphs
    main problem: variables projected out, i.e., do not appear in count

Conclusion

- gave dual version to classical theorem by Lovász
- generalizations
  - works for every finite set of nonisomorphic graphs
  - "primitive positive formulas"/"conjunctive queries", arbitrary finite structures instead of graphs
    main problem: variables projected out, i.e., do not appear in count
- give trichotomy for counting answers to unions of conjunctive queries, problem in database theory (PODS 2016)

Conclusion

- gave dual version to classical theorem by Lovász
- generalizations
  - works for every finite set of nonisomorphic graphs
  - "primitive positive formulas"/"conjunctive queries", arbitrary finite structures instead of graphs
    main problem: variables projected out, i.e., do not appear in count
- give trichotomy for counting answers to unions of conjunctive queries, problem in database theory (PODS 2016)

Thank you for your attention!