The Complexity of Coverability in ν-Petri Nets

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Outline

ν-Petri nets (νPN)

Petri nets with data management and creation
(Rosa-Velardo and de Frutos-Escrig, 2008, 2011)

coverability

- decidable by classical backward coverability algorithm (Abdulla et al., 2000)
- dual view using downwards-closed sets (Lazić and S., 2015)

complexity νPN coverability is complete for double Ackermann ($\mathbf{F}_{\omega \cdot 2}$-complete)

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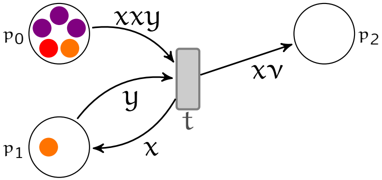
ν-Petri Nets

Tokens carry data from an infinite countable domain $\mathbb{D}$



Configurations in $(\mathbb{N}^P)^\circledast$: multisets of markings

$$\left[\begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}3\\0\\0\end{pmatrix}\begin{pmatrix}1\\1\\0\end{pmatrix}\right]$$

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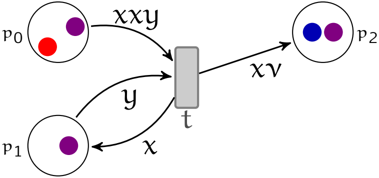

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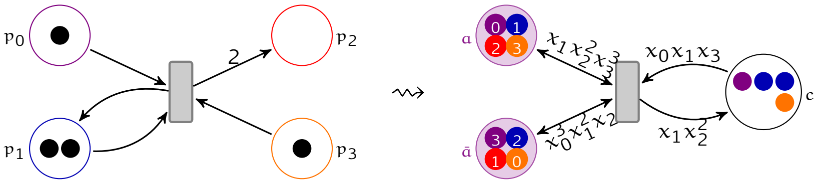
Petri Nets as ν-Petri Nets



- a and $\bar{a}$ are complementary addressing places
- c holds the actual token counts

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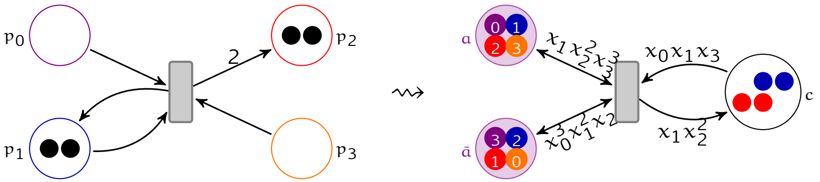

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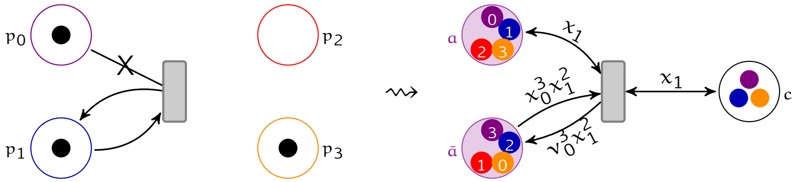
Reset Petri Nets as ν-Petri Nets



- a and $\bar{a}$ are complementary addressing places for active tokens
- c holds both the active and inactive tokens

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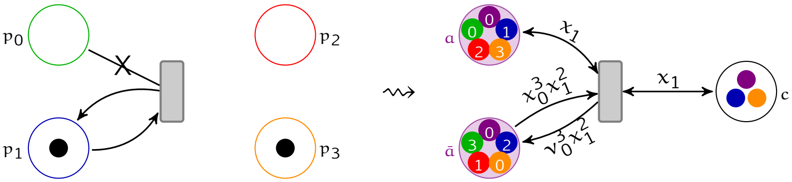

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Coverability Problem

verification of safety properties "nothing bad happens"

ordering of configurations by multiset embedding

$$[\mathbf{u}_1,\ldots,\mathbf{u}_n] \sqsubseteq [\mathbf{v}_1,\ldots,\mathbf{v}_p]$$

iff $\exists f : \{1,\ldots,n\} \to \{1,\ldots,p\}$ injective ,
$\forall 1 \leqslant i \leqslant n, u_i \leqslant v_{f(i)}$
Example:

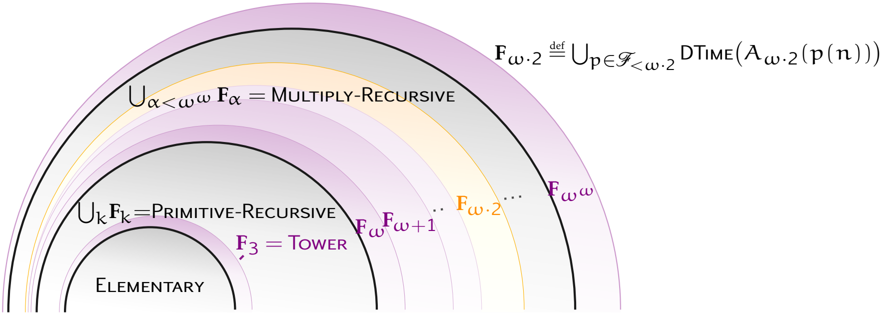
$$\left[\begin{pmatrix}1\\0\\1\end{pmatrix}\begin{pmatrix}3\\0\\0\end{pmatrix}\right] \sqsubseteq \left[\begin{pmatrix}10\\1\\0\end{pmatrix}\begin{pmatrix}1\\1\\0\end{pmatrix}\begin{pmatrix}2\\0\\3\end{pmatrix}\begin{pmatrix}0\\1\\1\end{pmatrix}\right]$$

input a νPN, a source configuration src, and a "bad" configuration tgt

question $\exists m, \mathrm{tgt} \sqsubseteq m$ and $\mathrm{src} \to^* m$?

FAST-GROWING COMPLEXITY

(S., 2016)



- Ackermann: "Ackermannian in" $x \mapsto 2x$

$$A_1(x) \stackrel{\text{def}}{=} 2x \qquad A_{k+2}(x) \stackrel{\text{def}}{=} A_{k+1}^x(1) \qquad A_\omega(x) \stackrel{\text{def}}{=} A_{x+1}(x)$$

- double Ackermann: "Ackermannian in" $A_\omega(x)$

$$A_{\omega+k+1}(x) \stackrel{\text{def}}{=} A_{\omega+k}^x(1) \qquad A_{\omega\cdot 2}(x) \stackrel{\text{def}}{=} A_{\omega+x+1}(x)$$

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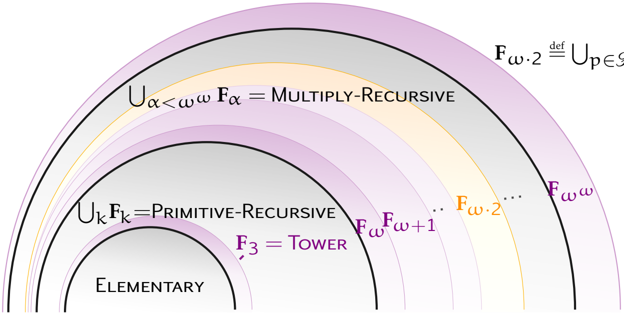

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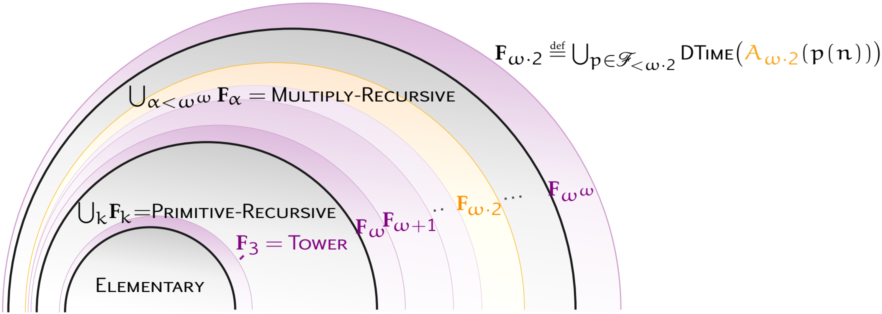

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Main Result

Theorem
Coverability in νPNs is $\mathbf{F}_{\omega\cdot 2}$-complete.

lower bound extends Lipton's "object-oriented" programming in Petri nets

- improves on the $\mathbf{F}_{\omega}$ lower bound of Schnoebelen (2010) for reset Petri nets
- basic block: Ackermann counters using Schnoebelen's construction
- pushed to double Ackermann: composition and iteration operations

Main Result

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upper bound analyses a dual view of the backward coverability algorithm (Lazić and S., 2015)

- the set of configurations not covering tgt is downwards-closed
- downwards-closed sets represented as finite sets of ideals
- exhibit a "star-monotone" sequence of ideals
- improves on the $\mathbf{F}_{\omega^\omega}$ upper bound of Rosa-Velardo (2014) for unordered data nets

Concluding Remarks

- first "natural" decision problem complete for $\mathbf{F}_{\omega\cdot 2}$
- ideals and downwards-closed sets as algorithmic tools
  - here, backward analysis (Lazić and S., 2015)
  - forward analysis (Finkel and Goubault-Larrecq, 2009, 2012)
  - reachability in Petri nets (Leroux and S., 2015)
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  - piecewise testable separability (Goubault-Larrecq and S., 2016)

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ν-Petri Nets are Well-Structured

(Finkel and Schnoebelen, 2001; Abdulla et al., 2000)

1. $((\mathbb{N}^P)^\circledast, \sqsubseteq)$ is a *well-quasi-order* (wqo), which entails

   finite bad sequences any sequence $m_0, m_1, m_2, \ldots$ with $\forall i < j, m_i \not\sqsubseteq m_j$, is finite

   finite basis property any upwards-closed subset U has a finite basis B such that $U = \uparrow B$

   ascending chain property all the ascending chains $U_0 \subsetneq U_1 \subsetneq U_2 \subsetneq \cdots$ of upwards-closed subsets are finite

2. *compatibility*: if $m_1 \sqsubseteq m'_1$ and $m_1 \to m_2$, then there exists m'_2, $m_2 \sqsubseteq m'_2$ and $m'_1 \to m'_2$

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"Classical" Backward Coverability

(Abdulla et al., 2000)

compute $U_* \stackrel{\text{def}}{=} \bigcup_k U_k$
where
$$U_k \stackrel{\text{def}}{=} \{m' \mid \exists m \sqsupseteq tgt, m' \to^{\leqslant k} m\}$$

initially $U_0 \stackrel{\text{def}}{=} \uparrow tgt$

step $U_{k+1} \stackrel{\text{def}}{=} Pre_\exists(U_k) \cup U_k$
where
$$Pre_\exists(S) \stackrel{\text{def}}{=} \{m \mid \exists s \in S, m \to s\}$$

representation of upwards-closed subsets U through their minimal elements thanks to finite basis property

termination guaranteed by ascending chain property

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(Lazić and S., 2015)

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representation of downwards-closed subsets D through finite representations of their *ideal decompositions* (next slide)

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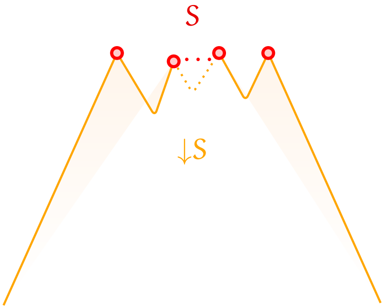
Ideal Decompositions for a wqo $(X, \leqslant)$

(Bonnet, 1975; Finkel and Goubault-Larrecq, 2009)

- Downward closure
  $\downarrow S \stackrel{\text{def}}{=} \{x \in X \mid \exists s \in S \,.\, x \leqslant s\}$
- Ideal I
  downwards-closed, non-empty and directed



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  $\forall x, y \in I, \exists z. x \leqslant z$ and $y \leqslant z$
- Examples
  - $\downarrow x \in \mathrm{Idl}(X)$ for any x in X
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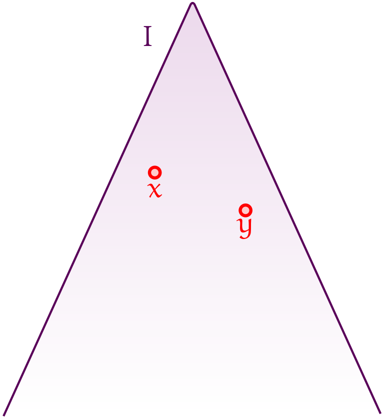

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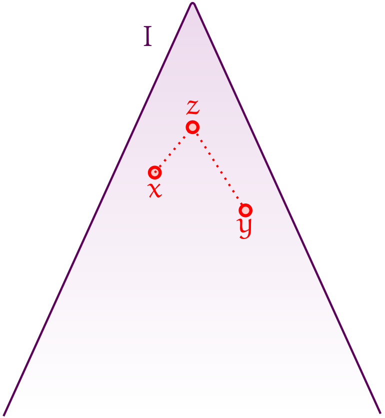

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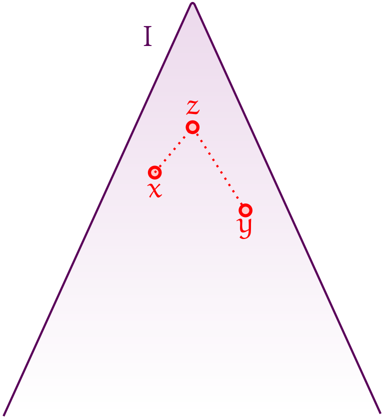

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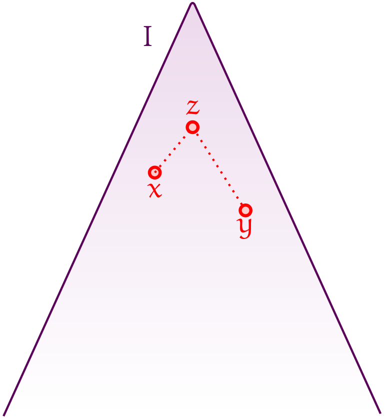

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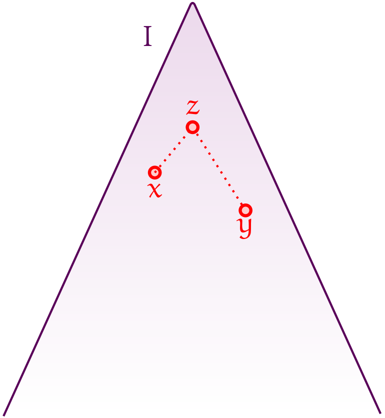
  $\downarrow S \stackrel{\text{def}}{=} \{x \in X \mid \exists s \in S \,.\, x \leqslant s\}$
- Ideal I
  downwards-closed, non-empty and directed:
  $\forall x, y \in I, \exists z. x \leqslant z$ and $y \leqslant z$
- Examples
  - $\downarrow x \in \mathrm{Idl}(X)$ for any x in X
  - $\mathbb{N} \in \mathrm{Idl}(\mathbb{N})$
  - $D^{\circledast} \in \mathrm{Idl}(X^{\circledast})$ for any $D \subseteq X$ downwards-closed



Ideal Decompositions for a wqo $(X, \leqslant)$

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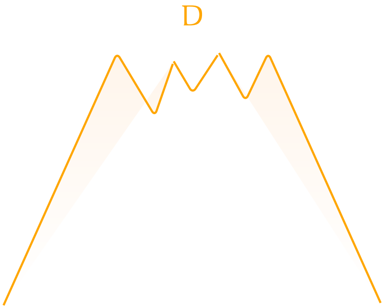

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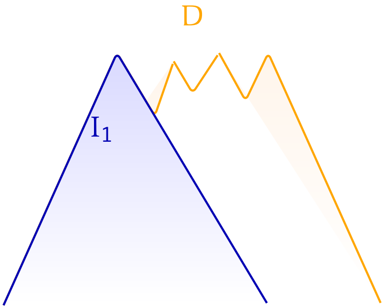

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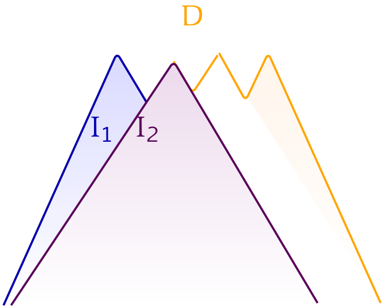

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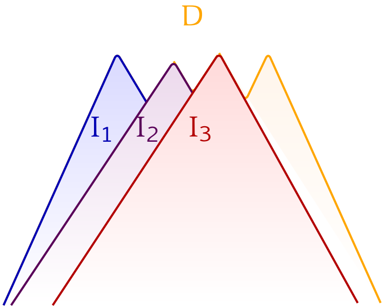

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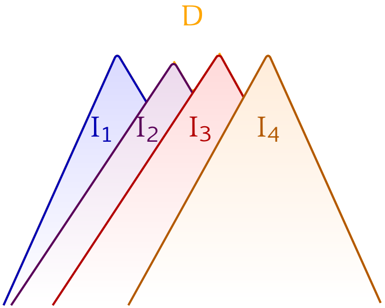

Effective Ideal Representations

(Finkel and Goubault-Larrecq, 2009; Goubault-Larrecq et al., 2016)

- extended markings:

$$\mathrm{Idl}(\mathbb{N}^P) = \{{\downarrow}\mathbf{u} \mid \mathbf{u} \in \mathbb{N}_\omega^P\}$$

where $\mathbb{N}_\omega^P \stackrel{\text{def}}{=} (\mathbb{N} \cup \{\omega\})^P$

- extended configurations:

$$\mathrm{Idl}((\mathbb{N}^P)^\circledast) = \{{\downarrow}(B,S) \mid B \in (\mathbb{N}_\omega^P)^\circledast, S \subseteq_f \mathbb{N}_\omega^P\}$$

- where $m \sqsubseteq (B,S)$ iff $\exists m' \in S^\circledast, m \sqsubseteq B \oplus m'$

- (B,S) is *reduced* iff S is an antichain and $\forall \mathbf{u} \in \mathrm{Support}(B), \forall \mathbf{v} \in S, \mathbf{u} \not\leq \mathbf{v}$

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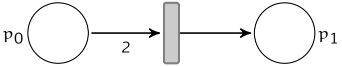
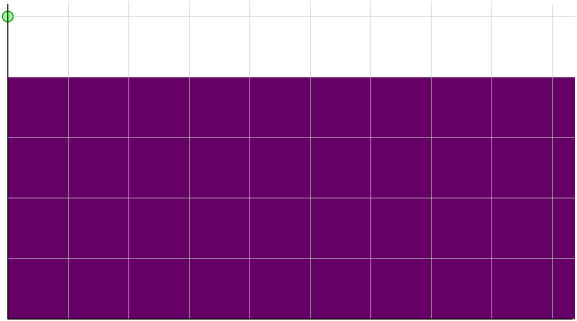
Dual Backward Coverability: Example



$\mathrm{tgt} = (0, 5)$



$$D_0 = \downarrow(\omega, 4)$$

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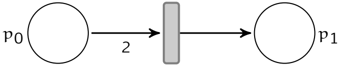

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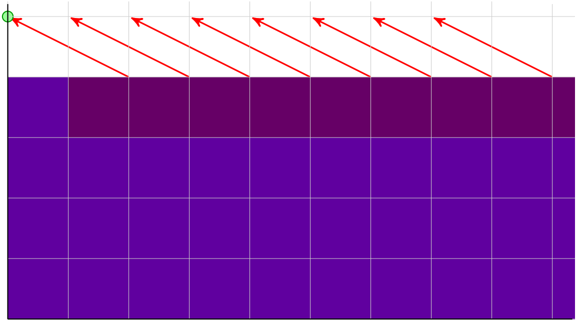
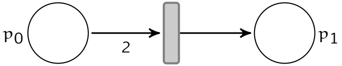


$$D_1 = \downarrow(1, 4) \cup \downarrow(\omega, 3)$$

Dual Backward Coverability: Example



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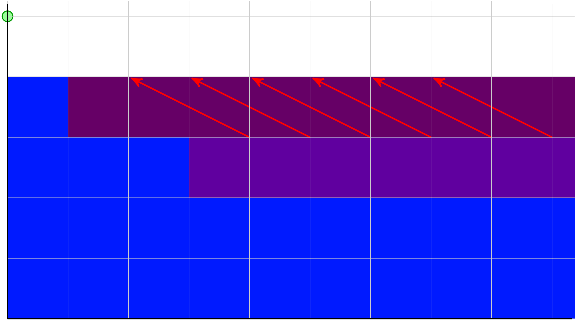
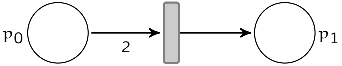


$$D_2 = \downarrow(1,4) \cup \downarrow(3,3) \cup \downarrow(\omega,2)$$

Dual Backward Coverability: Example



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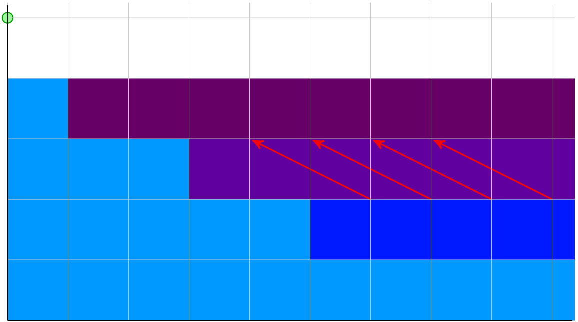
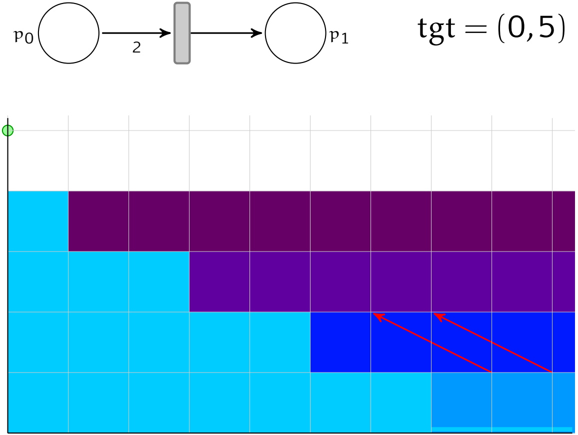


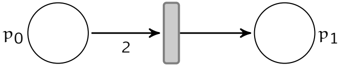
$$D_3 = \downarrow(1,4) \cup \downarrow(3,3) \cup \downarrow(5,2) \cup \downarrow(\omega,1)$$

Dual Backward Coverability: Example



$$D_4 = \downarrow(1,4) \cup \downarrow(3,3) \cup \downarrow(5,2) \cup \downarrow(7,1) \cup \downarrow(\omega,0)$$

Dual Backward Coverability: Example



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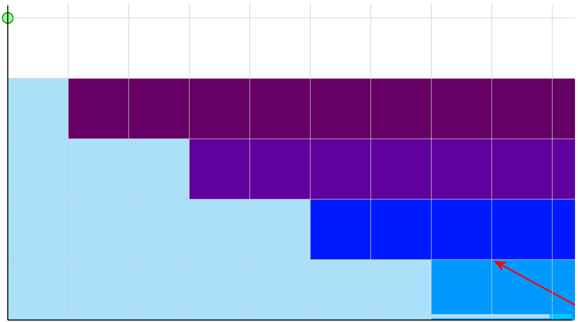

$$D_5 = \downarrow(1,4) \cup \downarrow(3,3) \cup \downarrow(5,2) \cup \downarrow(7,1) \cup \downarrow(9,0) = D_*$$

Controlled Sequences

- consider a *norm* $\|.\| : X \to \mathbb{N}$ with $\forall n, X_{\leqslant n} \stackrel{\text{def}}{=} \{x \in X \mid \|x\| \leqslant n\}$ finite:

$$\|\mathbf{u}\| \stackrel{\text{def}}{=} \max_{p \in P \mid \mathbf{u}(p) < \omega} \mathbf{u}(p) \qquad \text{for } \mathbf{u} \in \mathbb{N}^P_\omega$$

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- consider a *control function* $g : \mathbb{N} \to \mathbb{N}$ strictly monotone and an *initial norm* $n \in \mathbb{N}$

- a sequence $x_0, x_1, \ldots$ of elements of X is (g,n)-controlled if $\forall i, \|x_i\| \leqslant g^i(n)$

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Length Function Theorems (1/3)

(Figueira et al., 2011; S. and Schnoebelen, 2012)

Fact (Length Function Theorem for Bad Sequences in $\mathbb{N}_\omega^P$)

Let $n > 0$*. Any* (g,n)*-controlled bad sequence* $\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_\ell$ *of extended markings in* $(\mathbb{N}_\omega^P, \leqslant)$ *has length at most "Ackermannian in"* $g(\max(n, |P|))$*.*

Length Function Theorems (2/3)

(Lazić and S., 2015)

- consider a descending chain $D_0 \supsetneq D_1 \supsetneq \cdots \supsetneq D_\ell$
- extract at each step $0 \leqslant k < \ell$ a *proper ideal* I_k from the canonical decomposition of D_k, s.t. $I_k \not\subseteq D_{k+1}$
- *bad sequence* of proper ideals $I_0, I_1, \ldots, I_{\ell-1}$
- in particular, for descending chains $\downarrow S_0 \supsetneq \downarrow S_1 \supsetneq \cdots \supsetneq \downarrow S_\ell$ of antichains

Corollary (Length Function Theorem for Hoare-Descending Chains over $\mathbb{N}_\omega^P$)

Let $n > 0$. Any (g, n)-controlled descending chain $\downarrow S_0 \supsetneq \downarrow S_1 \supsetneq \cdots \supsetneq \downarrow S_\ell$ of antichains of $(\mathbb{N}_\omega^P, \leqslant)$ has length at most "Ackermannian in" $g(\max(n, |P|))$.

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COROLLARY (LENGTH FUNCTION THEOREM FOR HOARE-DESCENDING CHAINS OVER $\mathbb{N}_\omega^P$)

Let $n > 0$. Any (g,n)-controlled descending chain $\downarrow S_0 \supsetneq \downarrow S_1 \supsetneq \cdots \supsetneq \downarrow S_\ell$ of antichains of $(\mathbb{N}_\omega^P, \leqslant)$ has length at most "Ackermannian in" $g(\max(n, |P|))$.

Length Function Theorems (3/3)

- a descending chain $D_0 \supsetneq D_1 \supsetneq \cdots \supsetneq D_\ell$ over $(\mathbb{N}^P)^\circledast$ is *star-monotone* if $\forall 0 \leqslant k < \ell - 1, \forall I_{k+1} = \downarrow(B_{k+1}, S_{k+1})$ proper ideal from the canonical decomposition of D_{k+1}, $\exists I_k = \downarrow(B_k, S_k)$ proper ideal from the canonical decomposition of D_k s.t. $\downarrow S_{k+1} \subseteq \downarrow S_k$

Theorem (Length Function Theorem for Star-Monotone Descending Chains over $(\mathbb{N}^P_\omega)^\circledast$)

Let $n > 0$. Any strongly (g, n)-controlled star-monotone descending chain $D_0 \supsetneq D_1 \supsetneq \cdots \supsetneq D_\ell$ of configurations in $(\mathbb{N}^P_\omega)^\circledast$ has length at most "double Ackermannian in" $g(\max(n, |P|))$.

Wrapping Up

Lemma (Strong Control for νPNs)

The descending chain computed by the backward algorithm for a νPN $\mathcal{N}$ and target tgt is strongly (g,n)-controlled for $g(x) \stackrel{def}{=} x + |\mathcal{N}|$ and $n \stackrel{def}{=} \|\mathrm{tgt}\|$.

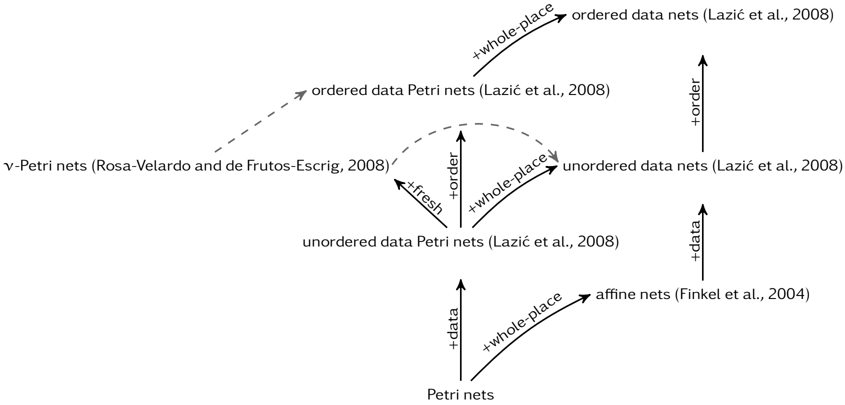
Lemma (νPN Descending Chains are Star-Monotone)

The descending chains computed by the backward coverability algorithm for νPNs are star-monotone.

Theorem (Upper Bound)

The coverability problem for νPNs is in $\mathbf{F}_{\omega\cdot 2}$.

Taxonomy of Petri Net Extensions



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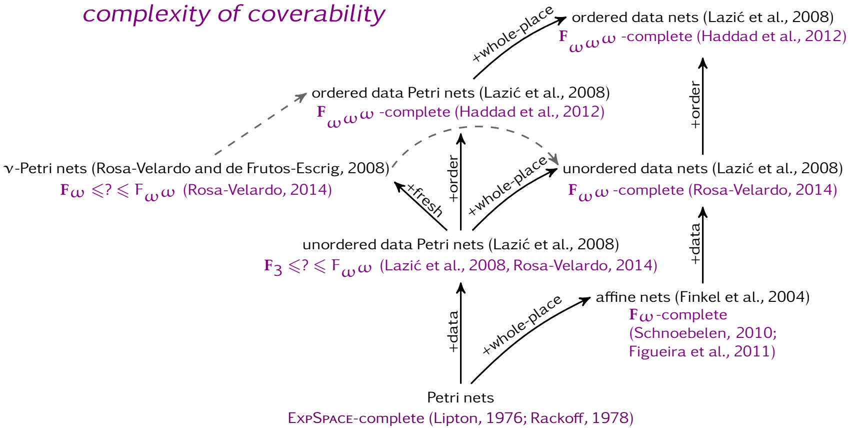
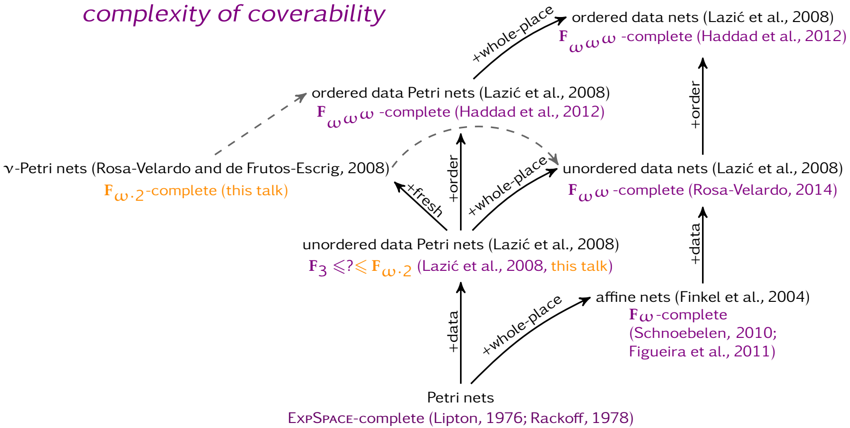


Taxonomy of Petri Net Extensions

complexity of coverability



Polyadic ν-Petri Nets

(Rosa-Velardo and Martos-Salgado, 2012)

- hold *tuples* of tokens in places
- equivalent to the full π-*calculus*
- model of *dynamic* database systems with existential positive guards
- *undecidable* coverability