Preservation and Decomposition Theorems for Bounded Degree Structures

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Aim:
Study complexity of computational problems for FO (+ extensions) on restricted classes of structures.
Introduction

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Study complexity of computational problems for FO (+ extensions) on restricted classes of structures.

Non-elementary complexity even on finite unranked forests

On input of $\varphi \in \text{FO}$, compute an equivalent

- Gaifman normal form (Dawar, Grohe, Kreutzer, Schweikardt 2007)
- Feferman-Vaught decomposition 
- existential FO-sentence* (if $\varphi$ is preserved under extensions)
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Elementary algorithms on classes of structures of bounded degree

On input of $\varphi \in \text{FO}$, compute an equivalent

- Gaifman normal form $(H., Kuske, Schweikardt 2013)$
- Feferman-Vaught decomposition $(Harwath, H., Schweikardt 2014)$
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  \[*\text{if } \varphi \text{ is preserved under extensions}\]

Lower bounds: 3-fold exponential.
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Lower bounds: 3-fold exponential.

Recall: $\text{FO+MOD}_m := \text{FO} + \text{modulo } m$ counting quantifiers

$$\exists \equiv_p \mod m y \ \psi(\bar{x}, y)$$
for all remainders $p \in [0, m-1]$
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Elementary algorithms on classes of structures of bounded degree

On input of $\varphi \in \text{FO+MOD}_m$, compute an equivalent

- Feferman-Vaught decomposition \hspace{1cm} \text{(Harwath, H., Schweikardt 2014)}
- Modulo normal form \hspace{1cm} \text{(new!)}

Lower bounds: 3-fold exponential.

Recall: \hspace{1cm} \text{FO+MOD}_m := \text{FO} + \text{modulo } m \text{ counting quantifiers}

$$\exists \equiv p \mod m y \ \psi(x, y)$$ \hspace{1cm} \text{for all remainders } p \in [0, m-1]
We fix a finite relational signature $\sigma$. 

Classes of structures of bounded degree

For each $d \geq 0$ $C_d := \{ \sigma$-structure $A : \text{maximum degree of } G(A) \leq d \}$
We fix a finite relational signature $\sigma$.

**Recall:**

**Gaifman graph** of a $\sigma$-structure $A$

$$G(A) := (A, E)$$

such that for all $a, b \in A$ there is an edge between $a$ and $b$

$\iff$ ex. $R \in \sigma$ and $\overline{t} \in R^A$ with $a, b \in \overline{t}$. 
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**Classes of structures of bounded degree**

For each $d \geq 0$

$$C^d := \{ \sigma\text{-structure } A : \text{maximum degree of } G(A) \leq d \}$$
We fix a finite relational signature $\sigma$.

Recall: 
**Gaifman graph** of a $\sigma$-structure $\mathcal{A}$

$$G(\mathcal{A}) := (A, E)$$

such that for all $a, b \in A$ there is an edge between $a$ and $b$ if and only if there exists $R \in \sigma$ and $t \in R^A$ with $a, b \in t$.

**Classes of structures of bounded degree**

For each $d \geq 0$

$$\mathcal{C}_d := \{\sigma\text{-structure } \mathcal{A} : \text{maximum degree of } G(\mathcal{A}) \leq d\}$$
Feferman-Vaught decompositions

Determine the theory of a composed structure by the theories of its component structures.
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Compositions:
Disjoint unions, cartesian products . . .
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Disjoint unions, cartesian products . . .

Applications:
▶ decidability results
▶ model- and satisfiability checking
▶ ingredient in other proofs (e.g., Gaifman’s theorem)
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**Algorithmic:**
Decompose formula into Boolean combination of formulas that speak about component structures.
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Disjoint unions, cartesian products . . .

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Decompose formula into Boolean combination of formulas that speak about component structures.
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Assume: \( \sigma \) has unary relation symbols \( \text{RED, BLUE,} \ldots \)
Feferman-Vaught decompositions

Assume: $\sigma$ has unary relation symbols $\text{RED}, \text{BLUE}, \ldots$

- $\sigma$-structure $\mathcal{A}$ is disjunctly colored $\iff$
  $\text{RED}^\mathcal{A}, \text{BLUE}^\mathcal{A}, \ldots$ is a partition of $A$ and
  all nodes adjacent in $G(\mathcal{A})$ have same color.
Feferman-Vaught decompositions

Assume: \( \sigma \) has unary relation symbols \( \text{RED}, \text{BLUE}, \ldots \)

- \( \sigma \)-structure \( \mathcal{A} \) is disjointly colored: \( \iff \)
  - \( \text{RED}^\mathcal{A}, \text{BLUE}^\mathcal{A}, \ldots \) is a partition of \( A \) and
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- $\varphi \in \text{FO+MOD}_m$ is **monochrome**: $\iff$
  
  ex. $K \in \{\text{RED, BLUE, ...}\}$ such that for all disjointly colored $\mathcal{A}$,
  
  $\mathcal{A} \models \varphi \iff \mathcal{A}|_{K^\mathcal{A}} \models \varphi$
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- $\varphi(\bar{x}) \in \text{FO} + \text{MOD}_m$ is monochrome $\iff$

  ex. $K \in \{\text{RED}, \text{BLUE}, \ldots\}$ such that for all disjointly colored $\mathcal{A}$,

  $$\mathcal{A} \models \varphi[\bar{a}] \iff (\mathcal{A}|_K \models \varphi[\bar{a}] \text{ and } \bar{a} \subseteq K^\mathcal{A})$$
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**Theorem** *(Harwath, H., Schweikardt 2014)*

For each \( \varphi(\overline{x}) \in \text{FO} \), a Feferman-Vaught decomposition, i.e., a Boolean combination of monochrome FO-formulas which is equivalent to \( \varphi(\overline{x}) \) on all disjointly colored \( \mathcal{A} \in \mathcal{C}_d \) can be computed in time \( 3\text{-exp}(\|\varphi\|) \).
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For each \( \varphi(\overline{x}) \in \text{FO} \), a Feferman-Vaught decomposition,
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can be computed in time \( 3\text{-exp}(\|\varphi\|) \).

*This is optimal.*
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Assume: \( \sigma \) has unary relation symbols \( \text{RED}, \text{BLUE}, \ldots \)

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- \( \varphi(\overline{x}) \in \text{FO+MOD}_m \) is monochrome : \( \iff \)
  ex. \( K \in \{ \text{RED}, \text{BLUE}, \ldots \} \) such that for all disjointly colored \( \mathcal{A}, \)
  \[ \mathcal{A} \models \varphi[\overline{a}] \iff (\mathcal{A}_{|K} \models \varphi[\overline{a}] \text{ and } \overline{a} \subseteq K^\mathcal{A}) \]

Theorem \( \text{(new!)} \)

For each \( \varphi(\overline{x}) \in \text{FO+MOD}_m \), a Feferman-Vaught decomposition, i.e., a Boolean combination of monochrome \( \text{FO+MOD}_m \)-formulas
which is equivalent to \( \varphi(\overline{x}) \) on all disjointly colored \( \mathcal{A} \in \mathcal{C}_d \)
can be computed in time \( 4\text{-exp}(\|\varphi\|) \).
Proof Sketch: Preliminaries

Let $\mathcal{A}$ be a $\sigma$-structure and $c \in \mathcal{A}$.

- $r$-neighbourhood $N^\mathcal{A}_r(c) := \{ b \in \mathcal{A} : dist^\mathcal{A}(b, c) \leq r \}$
Proof Sketch: Preliminaries

Let $\mathcal{A}$ be a $\sigma$-structure and $c \in A$.

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- $c$ realises $r$-type $\tau : \iff \mathcal{N}_r^\mathcal{A}(c) \cong \tau$
Proof Sketch: Preliminaries

Let $\mathcal{A}$ be a $\sigma$-structure and $\bar{c} \in A^n$.

- $r$-neighbourhood $N_r^A(\bar{c}) := \{ b \in A : \text{dist}^A(b, \bar{c}) \leq r \}$

- $r$-type $N_r^A(\bar{c}) := (\mathcal{A}|_{N_r^A(\bar{c})}, \bar{c})$

- $\bar{c}$ realises $r$-type $\tau : \iff N_r^A(\bar{c}) \cong \tau$
Proof Sketch for FO

Problem

Input: FO-formula $\varphi(\overline{x})$

Output: Boolean combination of monochrome FO-formulas that is $c_d$-equivalent to $\varphi(\overline{x})$
Proof Sketch for FO

Problem

- **Input**: FO-formula \( \varphi(x) \)
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**Step 1**: Construct \( \mathcal{C}_d \)-equivalent Hanf normal form \( \psi(x) \) for \( \varphi(x) \)
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Step 1: Construct $\mathcal{C}_d$-equivalent Hanf normal form $\psi(\overline{x})$ for $\varphi(\overline{x})$

Hanf-formula:

$$\exists \geq k y \text{sph}_\tau(\overline{x}, y)$$

where $\mathcal{A} \models \text{sph}_\tau[\overline{a}, b] \iff \mathcal{N}^\mathcal{A}_r(\overline{a}, b) \cong \tau.$
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Theorem

(Bollig, Kuske 2012)

For each $\varphi(\overline{x}) \in \text{FO}$, a $\mathcal{C}_d$-equivalent Hanf normal form, i.e., a Boolean combination of Hanf-formulas, can be computed in time $3\text{-exp}(\|\varphi\|)$. 

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Theorem (Bollig, Kuske 2012)

For each $\varphi(\overline{x}) \in FO$, a $C_d$-equivalent Hanf normal form, i.e., a Boolean combination of Hanf-formulas, can be computed in time $3\text{-exp}(\|\varphi\|)$. This is optimal.
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**Input:** FO-formula $\varphi(\overline{x})$

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Step 1: Construct $c_d$-equivalent Hanf normal form $\psi(\overline{x})$ for $\varphi(\overline{x})$
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Step 1: Construct $\mathcal{C}_d$-equivalent Hanf normal form $\psi(x)$ for $\varphi(x)$

Case 1: $\tau$ is not disjointly colored

Step 2: Decompose each Hanf-formula $\alpha(x) := \exists^\geq k y \ sph_\tau(x, y)$ in $\psi(x)$
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<thead>
<tr>
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</tr>
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Step 2: Decompose each Hanf-formula $\alpha(x) := \exists^{\geq k} y \ sph_\tau(x, y)$ in $\psi(x)$

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$\rightarrow \alpha$ is unsatisfiable in disjointly colored structures
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$\longrightarrow \alpha$ is unsatisfiable in disjointly colored structures
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Input: FO-formula $\varphi(\overline{x})$

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Case 1: $\tau$ is not disjointly colored

$\rightarrow \alpha$ is unsatisfiable in disjointly colored structures

Case 2: $\tau$ is disjointly colored

\begin{center}
\begin{tikzpicture}
  \fill[red] (0,0) circle (1cm);
  \fill[blue] (2,0) circle (1cm);
\end{tikzpicture}
\end{center}
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Case 1: $\tau$ is not disjointly colored

$\longrightarrow \alpha$ is unsatisfiable in disjointly colored structures

Case 2: $\tau$ is disjointly colored

$\longrightarrow \text{Replace } \alpha \text{ by conjunction of monochrome Hanf-formulas}$
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Case 1: $\tau$ is not disjointly colored

$\rightarrow \alpha$ is unsatisfiable in disjointly colored structures

Case 2: $\tau$ is disjointly colored

$\rightarrow$ Replace $\alpha$ by conjunction of monochrome Hanf-formulas
Proof Sketch for $\text{FO+MOD}_m$

Problem

Input: $\text{FO+MOD}_m$-formula $\varphi(\overline{x})$

Output: Boolean combination of monochrome $\text{FO+MOD}_m$-formulas that is $c_d$-equivalent to $\varphi(\overline{x})$
Proof Sketch for FO+MOD$_m$

**Problem**

**Input:** FO+MOD$_m$-formula $\varphi(x)$

**Output:** Boolean combination of monochrome FO+MOD$_m$-formulas that is $\mathcal{C}_d$-equivalent to $\varphi(x)$

**Step 1:** Construct $\mathcal{C}_d$-equivalent **Modulo normal form** $\psi(x)$ for $\varphi(x)$
Proof Sketch for FO+MOD$_m$

Problem

Input: FO+MOD$_m$-formula $\varphi(\overline{x})$

Output: Boolean combination of monochrome FO+MOD$_m$-formulas that is $\mathcal{C}_d$-equivalent to $\varphi(\overline{x})$

Step 1: Construct $\mathcal{C}_d$-equivalent Modulo normal form $\psi(\overline{x})$ for $\varphi(\overline{x})$

Modulo-formula:

$$\exists \equiv^p \text{mod} m y \ \text{sph}_\tau(\overline{x}, y)$$

where $\mathcal{A} \models \text{sph}_\tau[\overline{a}, b] \iff \mathcal{N}_r\mathcal{A}(\overline{a}, b) \cong \tau$. 
Proof Sketch for FO+MOD\textsubscript{m}

Problem

Input: FO+MOD\textsubscript{m}-formula \( \varphi(\overline{x}) \)

Output: Boolean combination of monochrome FO+MOD\textsubscript{m}-formulas that is \( \mathcal{C}_d \)-equivalent to \( \varphi(\overline{x}) \)

Step 1: Construct \( \mathcal{C}_d \)-equivalent Modulo normal form \( \psi(\overline{x}) \) for \( \varphi(\overline{x}) \)

Modulo-formula:

\[
\exists \equiv^{p \mod m} y \ sph_{\tau}(\overline{x}, y)
\]

where \( A \models sph_{\tau}[\overline{a}, b] \iff N_{r}^{A}(\overline{a}, b) \cong \tau. \)

Theorem \textit{(new!)}

For each \( \varphi(\overline{x}) \in FO+MOD_{m} \), a \( \mathcal{C}_d \)-equivalent Modulo normal form, i.e., a Boolean combination of Hanf-formulas and Modulo-formulas, can be computed in time \( 4\text{-exp}(\|\varphi\|) \).
Proof Sketch for FO+MOD\(_m\)

Problem

Input: FO+MOD\(_m\)-formula \(\varphi(\bar{x})\)

Output: Boolean combination of monochrome FO+MOD\(_m\)-formulas that is \(\mathcal{C}_d\)-equivalent to \(\varphi(\bar{x})\)

Step 1: Construct \(\mathcal{C}_d\)-equivalent Modulo normal form \(\psi(\bar{x})\) for \(\varphi(\bar{x})\)

Modulo-formula:
\[
\exists y \equiv \exists p \mod m y \text{ sph}_\tau(\bar{x}, y)
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where \(\mathcal{A} \models \text{sph}_\tau[\bar{a}, b] \iff \mathcal{N}_r^\mathcal{A}(\bar{a}, b) \simeq \tau\).

Theorem (new!)

For each \(\varphi(\bar{x}) \in \text{FO+MOD}_m\), a \(\mathcal{C}_d\)-equivalent Modulo normal form, i.e., a Boolean combination of Hanf-formulas and Modulo-formulas, can be computed in time \(4\text{-exp}(\|\varphi\|)\).
Proof Sketch for FO+MOD$_m$

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Input: FO+MOD$_m$-formula $\varphi(\bar{x})$

Output: Boolean combination of monochrome FO+MOD$_m$-formulas that is $c_d$-equivalent to $\varphi(\bar{x})$

Step 1: Construct $c_d$-equivalent Modulo normal form $\psi(\bar{x})$ for $\varphi(\bar{x})$ ✓

Step 2: Decompose each Modulo- or Hanf-formula in $\psi(\bar{x})$
Proof Sketch for FO+MOD\textsubscript{m}

**Problem**

- **Input:** FO+MOD\textsubscript{m}-formula \( \varphi(\overline{x}) \)
- **Output:** Boolean combination of monochrome FO+MOD\textsubscript{m}-formulas that is \( c_d \)-equivalent to \( \varphi(\overline{x}) \)

**Step 1:** Construct \( c_d \)-equivalent Modulo normal form \( \psi(\overline{x}) \) for \( \varphi(\overline{x}) \) ✔

**Step 2:** Decompose each Modulo- or Hanf-formula in \( \psi(\overline{x}) \) ✔

\[ \ldots \text{suitable adaptation of Step 2 for FO} \]
Main Results

Upper Bounds

There are algorithms that, on input of \( \varphi \in FO \), compute in time a

<table>
<thead>
<tr>
<th>3-exp(|\varphi|)</th>
<th>Hanf normal form</th>
<th>(Bollig, Kuske 2012)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3-exp(|\varphi|)</td>
<td>Feferman-Vaught decomposition</td>
<td></td>
</tr>
</tbody>
</table>

that is equivalent to \( \varphi \) on \( \mathcal{C}_d \).
Main Results

Upper Bounds

There are algorithms that, on input of $\varphi \in \text{FO}$, compute in

<table>
<thead>
<tr>
<th>time</th>
<th>a</th>
</tr>
</thead>
<tbody>
<tr>
<td>3-exp($|\varphi|$)</td>
<td>Hanf normal form</td>
</tr>
<tr>
<td>3-exp($|\varphi|$)</td>
<td>Feferman-Vaught decomposition</td>
</tr>
<tr>
<td>5-exp($|\varphi|$)</td>
<td>existential FO-sentence*</td>
</tr>
<tr>
<td>4-exp($|\varphi|$)</td>
<td>existential-positive FO-sentence*</td>
</tr>
</tbody>
</table>

\* if $\varphi$ is preserved under extensions/homomorphisms on $\mathcal{C}_d$

that is equivalent to $\varphi$ on $\mathcal{C}_d$. 
# Main Results

## Upper Bounds

There are algorithms that, on input of $\varphi \in \text{FO}$, compute in time a

<table>
<thead>
<tr>
<th>time</th>
<th>3-exp($|\varphi|$)</th>
<th>3-exp($|\varphi|$)</th>
<th>5-exp($|\varphi|$)</th>
<th>4-exp($|\varphi|$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>result</td>
<td>Hanf normal form</td>
<td>Feferman-Vaught decomposition</td>
<td>existential FO-sentence*</td>
<td>existential-positive FO-sentence*</td>
</tr>
</tbody>
</table>

that is equivalent to $\varphi$ on $\mathcal{C}_d$.

*(Bollig, Kuske 2012)*

## Lower bounds

3-fold exponential for FO.
# Main Results

## Upper Bounds

There are algorithms that, on input of $\varphi \in \text{FO (FO+MOD}_m\text{)},$ compute in time a

<table>
<thead>
<tr>
<th>Time</th>
<th>Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3(4)^{-\text{exp}(|\varphi|)}$</td>
<td>Hanf (Modulo) normal form (Bollig, Kuske 2012)</td>
</tr>
<tr>
<td>$3(4)^{-\text{exp}(|\varphi|)}$</td>
<td>Feferman-Vaught decomposition</td>
</tr>
<tr>
<td>$5(6)^{-\text{exp}(|\varphi|)}$</td>
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* if $\varphi$ is preserved under extensions/homomorphisms on $\mathcal{C}_d$ that is equivalent to $\varphi$ on $\mathcal{C}_d$.

## Lower bounds

3-fold exponential for FO.
Main Results

Upper Bounds

There are algorithms that, on input of $\varphi \in \text{FO (FO+MOD}_m)$, compute in time

<table>
<thead>
<tr>
<th>Expression</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3(4)$-$\exp(|\varphi|)$</td>
<td>Hanf (Modulo) normal form $(\text{Bollig, Kuske 2012})$</td>
</tr>
<tr>
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* if $\varphi$ is preserved under extensions/homomorphisms on $\mathcal{C}$

that is equivalent to $\varphi$ on $\mathcal{C} \subseteq \mathcal{C}_d$,
which is closed under disjoint unions and induced substructures.

Lower bounds

3-fold exponential for FO.
## Main Results

### Upper Bounds

There are algorithms that, on input of $\varphi \in \text{FO} (\text{FO+MOD}_m)$, compute in time $a^{3(4)^{\text{exp}(\|\varphi\|)}}$ Hanf (Modulo) normal form \((Bollig, Kuske 2012)\)

<table>
<thead>
<tr>
<th>Time</th>
<th>Transformation</th>
</tr>
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<tbody>
<tr>
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<td>Hanf (Modulo) normal form</td>
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### Lower bounds

3-fold exponential for FO.

### Ongoing work:

Close the gaps between upper and lower bounds.
Main Results

Upper Bounds

There are algorithms that, on input of $\varphi \in \text{FO (FO+MOD}_m)$, compute in time a

<table>
<thead>
<tr>
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</tr>
</thead>
<tbody>
<tr>
<td>$3(3)$-exp($|\varphi|$)</td>
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* if $\varphi$ is preserved under extensions/homomorphisms on $\mathcal{C}$

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Lower bounds

3-fold exponential for FO.

Ongoing work: Close the gaps between upper and lower bounds.
Thank you!

Upper Bounds

There are algorithms that, on input of $\varphi \in \text{FO (FO+MOD}_m)$, compute in time $a^{|\varphi|}$

| $3(3)$-exp($|\varphi|$) | Hanf (Modulo) normal form $(Bollig, Kuske 2012)$ |
|-------------------------|-----------------------------------------------|
| $3(3)$-exp($|\varphi|$) | Feferman-Vaught decomposition                  |
| $5(6)$-exp($|\varphi|$) | existential FO-sentence*                      |
| $4(4)$-exp($|\varphi|$) | existential-positive FO-sentence*             |

* if $\varphi$ is preserved under extensions/homomorphisms on $\mathcal{C}$

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Lower bounds

3-fold exponential for FO.

Ongoing work: Close the gaps between upper and lower bounds.