Decidability of Weak Logics with Deterministic Transitive Closure

Witold Charatonik\(^1\)  Emanuel Kieroński\(^1\)  Filip Mazowiecki\(^2\)

\(^1\)University of Wrocław  
\(^2\)University of Warsaw  

September 5, 2014
Basic facts about FO and FO$^2$

- FO undecidable (Church, Turing; 1930s)
- FO$^3$ undecidable (Kahr, Moore, Wang; 1959)
Basic facts about FO and FO$^2$

- FO undecidable (Church, Turing; 1930s)
- FO$^3$ undecidable (Kahr, Moore, Wang; 1959)
- FO$^2$ decidable
Basic facts about FO and FO<sup>2</sup>

- FO undecidable (Church, Turing; 1930s)
- FO<sup>3</sup> undecidable (Kahr, Moore, Wang; 1959)
- FO<sup>2</sup> decidable
- FO<sup>2</sup> exponential model property (Grädel, Kolaitis, Vardi; 1997), NExpTime-completeness

Decidability of Weak Logics with Deterministic Transitive Closure

Highlights 2014

September 5, 2014
Basic facts about FO and FO²

- FO undecidable (Church, Turing; 1930s)
- FO³ undecidable (Kahr, Moore, Wang; 1959)
- FO² decidable
- FO² exponential model property (Grädel, Kolaitis, Vardi; 1997), NExpTime-completeness
- FO² on trees is ExpSpace-complete (Benaim et al.; 2013).
Equivalence closure (EC)

- In general undecidable.
- EC of 1 or 2 binary relations – decidable (Kieroński et al.; 2012)
Equivalence closure (EC)

- In general undecidable.
- EC of 1 or 2 binary relations – decidable (Kieroński et al.; 2012)

Transitive closure (TC)

- In general undecidable.
- TC of 1 binary relation – ?
Equivalence closure (EC)

- In general undecidable.
- EC of 1 or 2 binary relations – decidable (Kieroński et al.; 2012)

Transitive closure (TC)

- In general undecidable.
- TC of 1 binary relation – ?
  partial results [Kieroński, Michaliszyn; 2012]; [Szwast, Tendera; 2013]
$E' \subseteq E$ such that $(a, b), (a, c) \in E \implies b = c$
$E' \subseteq E$ such that $(a, b), (a, c) \in E \implies b = c$

$\text{DTC}(E) = \overline{E} = \text{TC}(E')$
$E' \subseteq E$ such that $(a, b), (a, c) \in E \implies b = c$

$\text{DTC}(E) = \overline{E} = \text{TC}(E')$
$E' \subseteq E$ such that $(a, b), (a, c) \in E \implies b = c$

$\text{DTC}(E) = \overline{E} = \text{TC}(E')$

$E'$-relation
$E' \subseteq E$ such that $(a, b), (a, c) \in E \implies b = c$

$\text{DTC}(E) = \overline{E} = \text{TC}(E')$

$\overline{E}$-relation
Examples

- partial function $\forall xy (E(x, y) \rightarrow \overline{E}(x, y))$
Examples

- partial function $\forall xy(E(x, y) \rightarrow \overline{E}(x, y))$

- rooted tree
  - $\exists x(R(x) \land \forall y(R(y) \rightarrow x = y) \land \forall y \neg E(x, y))$ (unique root)
  - $\forall xy(R(x) \land \neg R(y) \rightarrow \overline{E}(y, x))$ (unique path to the root)

$E = \uparrow$, $\overline{E} = \uparrow^+$
Examples

- partial function $\forall xy (E(x, y) \rightarrow \overline{E}(x, y))$
- rooted tree
  - $\exists x (R(x) \land \forall y (R(y) \rightarrow x = y) \land \forall y \neg E(x, y))$ (unique root)
  - $\forall xy (R(x) \land \neg R(y) \rightarrow \overline{E}(y, x))$ (unique path to the root)

$(E = \uparrow, \overline{E} = \uparrow^+)$

- infinite structure $\forall x (\neg \overline{E}(x, x) \land \exists y \overline{E}(x, y))$
On ordered structures $\text{FO} + \text{DTC}$ captures $\text{LogSpace}$ (Immerman; 1987);
About DTC

- On ordered structures \( \text{FO} + \text{DTC} \) captures \( \text{LOGSPACE} \) (Immerman; 1987);
- \( \text{FO}^2 + \text{DTC} \) is undecidable (Grädel, Otto, Rosen; 1999);
About DTC

- On ordered structures $\text{FO} + \text{DTC}$ captures $\text{LOGSPACE}$ (Immerman; 1987);
- $\text{FO}^2 + \text{DTC}$ is undecidable (Grädel, Otto, Rosen; 1999);
- $\exists^* \forall^* + \text{DTC}^+(E)$ is $\text{NEXPSPACE}$-complete (Immerman et al.; 2004);
• On ordered structures $\mathit{FO} + \mathit{DTC}$ captures $\text{LogSpace}$ (Immerman; 1987);
• $\mathit{FO}^2 + \mathit{DTC}$ is undecidable (Grädel, Otto, Rosen; 1999);
• $\exists^* \forall^* + \mathit{DTC}^+ (E)$ is $\text{NExpTime}$-complete (Immerman et al.; 2004); Allowing $\mathit{DTC}^-$ leads to undecidability.
Our logic

$FO^2 + DTC(E)$

- unary predicates: $p, q, r, \ldots$
- 1 binary relation: $E$
- its Deterministic Transitive Closure: $\bar{E}$
- only two variables: $x, y$. 
Our logic

$\text{FO}^2 + \text{DTC}(E)$

- unary predicates: $p, q, r, \ldots$;
- 1 binary relation: $E$;
- its Deterministic Transitive Closure: $\overline{E}$;
- only two variables: $x, y$.

Comparing to $\exists^* \forall^* + \text{DTC}^+(E)$:
Our logic

$\text{FO}^2 + \text{DTC}(E)$

- unary predicates: $p, q, r, \ldots$;
- 1 binary relation: $E$;
- its Deterministic Transitive Closure: $\overline{E}$;
- only two variables: $x, y$.

Comparing to $\exists^* \forall^* + \text{DTC}^+(E)$:

- only two variables 😊
Our logic

$\text{FO}^2 + \text{DTC}(E)$

- unary predicates: $p, q, r, \ldots$;
- 1 binary relation: $E$;
- its Deterministic Transitive Closure: $\overline{E}$;
- only two variables: $x, y$.

Comparing to $\exists^* \forall^* + \text{DTC}^+(E)$:

- only two variables 😊
- negative occurrences of DTC 😊
Our logic

\[ \text{FO}^2 + \text{DTC}(E) \]

- unary predicates: \( p, q, r, \ldots \);
- 1 binary relation: \( E \);
- its Deterministic Transitive Closure: \( \overline{E} \);
- only two variables: \( x, y \).

Comparing to \( \exists^* \forall^* + \text{DTC}^+(E) \):

- only two variables
- negative occurrences of DTC
- unrestricted nested quantification
Theorem.

Satisfiability (finite satisfiability) problems for $\text{FO}^2 + \text{DTC}(E)$ are $\text{ExpSpace}$-complete.
Theorem.

Satisfiability (finite satisfiability) problems for $\text{FO}^2 + \text{DTC}(E)$ are $\text{ExpSpace}$-complete.

Remember?

$\text{FO}^2 + \text{DTC}$ is undecidable (Grädel, Otto, Rosen; 1999);
$E \cap \text{DTC}(E)$ is $\rightarrow$, nondeterministic edges are $\longrightarrow$. 

A typical model

Decidability of Weak Logics with Deterministic Transitive Closure

Highlights 2014

September 5, 2014 9 / 11
A typical model

\[ E \cap \text{DTC}(E) \] is \( \rightarrow \), nondeterministic edges are \( \longrightarrow \).

Cyclically rooted (blue)
A typical model

$E \cap \text{DTC}(E)$ is $\rightarrow$, nondeterministic edges are $\longrightarrow$.

Cyclically rooted (blue) Rooted (red, orange)
A typical model

$E \cap \text{DTC}(E)$ is $\rightarrow$, nondeterministic edges are $\rightarrow\rightarrow$.

Cyclically rooted (blue)  Rooted (red, orange)  Top unbounded (green)
Theorem.

Satisfiability (finite satisfiability) problems for $\text{FO}^2 + \text{DTC}(E)$ are $\text{ExpSpace}$-complete.
Theorem.

*Satisfiability (finite satisfiability) problems for $\text{FO}^2 + \text{DTC}(E)$ are $\text{ExpSpace}$-complete.*

If $\varphi$ is satisfiable then there is a model s.t.:
Theorem.

Satisfiability (finite satisfiability) problems for $\text{FO}^2 + \text{DTC}(E)$ are ExpSpace-complete.

If $\varphi$ is satisfiable then there is a model s.t.:

- Every tree is “small”
Theorem.

Satisfiability (finite satisfiability) problems for $\text{FO}^2 + \text{DTC}(E)$ are $\text{ExpSpace}$-complete.

If $\varphi$ is satisfiable then there is a model s.t.:

- Every tree is “small”
- The number of trees is “small”
Conclusions

- $\mathsf{FO}^2 + \mathsf{DTC}(E)$ is $\mathsf{ExpSpace}$-complete;
- its universal fragment is $\mathsf{NExpTime}$-complete;
- some undecidability results.