The Relations between Directed Tree-width & Co.

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joint work with

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Treewidth

Bags:
- treelike connected
- contain every vertex
- contain every edge
- \( v \in \text{bag}_1 \) and \( v \in \text{bag}_2 \) \( \Rightarrow \) \( v \) in all bags inbetween
- Treewidth of \( G \): maximum size of a bag in a minimum decomposition (minus 1)
Treewidth: Nice Properties

► A lot of different characterisations
  ► via cops and robber games
  ► elimination orderings
  ► inductive constructions

► Many algorithmic applications
► computing treewidth is NP-complete, but in PFT, good fast approximations
► good structural properties in connection to minors
Widths Generalising Treewidth to Digraphs

- directed tree-width
  - [Reed 99; Johnson, Robertson, Seymour, Thomas 01]
    - arboreal decompositions, computable in NP and FPT
    - linkage problems (NP-complete in general) are in XP: $k$-disjoint paths, Hamiltonian path, ... 
    - cops and robber game: can be made robber-monotone, but not cop-monotone
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- Kelly-width [Hunter, Kreutzer 07]
  - Kelly-decompositions, in NP and XP
  - elimination orderings
  - $\text{dtw}(G) < \text{Kw}(G)$ (dtw bounded in Kw, but not vice versa)
  - additionally to directed treewidth: parity games in XP, $L_\mu$-MC in FPT [Bojanczyk, Kreutzer, Dittmann 14]
  - monotonicity cost bounded? – an open question
Generalising treewidth to Digraphs

- **DAG-width**
  [Obdržálek 06; Berwanger, Dawar, Hunter, Kreutzer 06; BDHKO 12]
  - DAG decompositions, but can be super-polynomially big
  - deciding if $\text{dagw}(G) \leq k$ is \text{PSPACE}-complete.
  - $\text{dtw}(G) < \text{dagw}(G)$
  - additionally to directed treewidth: parity games in XP, $L_\mu$-MC in FPT [Bojanczyk, Kreutzer, Dittmann 14]
  - monotonicity cost bounded? – a partial answer,
    - which implies $\text{dagw}(G) \leq Kw(G)$
    - defined weak DAG-width:
      - computable in NP,
      - seems to be algorithmically as useful as DAG-width
Generalising treewidth to Digraphs

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- **D-width** [Safari 05]
  - D-decompositions, in NP
  - \( \text{dtw}(G) \leq \text{dw}(G) \), even \( \text{dtw}(G) < \text{dw}(G) \)
Generalising treewidth to Digraphs

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- **oriented tree-width** [discussed in the community, but not published]
The Scheme (Our Results in Red)

\[
\begin{align*}
\text{dtw} & < \text{cmdtw} < \text{DAG-w} \leq \text{K-w} \\
\text{rmdtw} & \parallel \\
\text{D-w} & < \text{otw}
\end{align*}
\]
The Scheme (Our Results in Red)

dtw \succ cmdtw \succ DAG-w \preceq K-w

example classes of graphs
The Scheme (Our Results in Red)

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*using weak DAG-width*
The Scheme (Our Results in Red)

using weak DAG-width

\[
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rmdtw & \parallel \\
& & D-w & < & otw
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Thank you!