Interval Temporal Logics and Equivalence Relations

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HIGHLIGHTS 2014
The effects/benefits of the addition of one or more equivalence relations to a logic have been already studied in various settings, including (fragments of) first-order logic, linear temporal logic, metric temporal logic, and interval temporal logic.

There exists a close relationship between interval temporal logics and fragments of first-order logic, that allows the transfer of results and logical tools (e.g., tableau systems) between them.
**Interval Temporal Logics**

- Interval temporal logics: an alternative approach to point-based temporal representation and reasoning.

  Truth of formulas is defined over intervals rather than points.

- Halpern and Shoham’s modal logic of intervals (HS)
  - HS features 12 modalities, one for each possible ordering of a pair of intervals (the so-called Allen’s relations);
  - decidability and expressiveness of HS fragments (restrictions to subsets of HS modalities) have been systematically studied in the last decade.

- Decidability and expressiveness depend on two crucial factors: the selected set of modalities and the class of linear orders on which they are interpreted.
IN THIS TALK

▶ We focus our attention on the satisfiability problem for some meaningful fragments of HS extended with one or more equivalence relations, interpreted over the class of finite linear orders: the interval logic of temporal neighborhood $\bar{A\bar{A}}$ (aka PNL), its metric extension MPNL, and $AB$. 
In this talk

- We focus our attention on the satisfiability problem for some meaningful fragments of HS extended with one or more equivalence relations, interpreted over the class of finite linear orders: the interval logic of temporal neighborhood $\bar{A} \bar{A}$ (aka PNL), its metric extension MPNL, and $AB$.

- The original contributions can be summarized as follows:
  - decidability ($\text{NEXPTIME-completeness}$) of PNL$\sim$ (the extension of PNL with one equivalence relation);
  - decidability ($\text{NEXPTIME-completeness}$) of MPNL$\sim$ (the extension of MPNL with one equivalence relation);
  - undecidability of $AB\sim_{1\sim2}$ (the extension of $AB$ with two equivalence relations).
2. (Metric) PNL

Syntax and Semantics of PNL
Previous results
Proof structure
(Metric) PNL
Introduction

(Metric) PNL

AB

Related work

Conclusions

SYNTAX AND SEMANTICS OF PNL

Formulas of PNL, built from Allen’s relations *meets* and *met by*, are recursively defined by the following grammar:

\[ \varphi ::= p \mid \neg \varphi \mid \varphi \lor \varphi \mid \langle A \rangle \varphi \mid \langle \bar{A} \rangle \varphi \]
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- **PNL ~**
  - We extend the language with a special propositional symbol ~ interpreted as an equivalence relation over the points of the domain.
  - An interval \([x, y]\) satisfies ~ if and only if \(x\) and \(y\) belong to the same equivalence class.
PREVIOUS RESULTS

- The satisfiability problem for PNL over finite linear orders is $NEXPTIME$-complete.
  - there is a polynomial reduction from the satisfiability problem for the two-variable fragment of first-order logic $FO^2[<]$ to the satisfiability problem for PNL, and vice versa;
  - $FO^2[<]$ is $NEXPTIME$-complete.


Decidibility of PNL~

Theorem

The satisfiability problem for PNL~ is decidable (NEXPTIME-complete) on the class of finite linear orders.
Decidability of PNL∼

Theorem

The satisfiability problem for PNL∼ is decidable (NEXPTIME-complete) on the class of finite linear orders.

The expressive completeness of PNL with respect to FO²[<] can be easily extended to PNL∼ and FO²[<,∼], and thus:

Corollary

FO²[<,∼] is decidable (NEXPTIME-complete) on the class of finite linear orders.
**PROOF STRUCTURE**

The proof is a combination of 3 lemmas:

1. the first one provides an (exponential) **upper bound** to the **cardinality** of each equivalence class in a minimal model;

2. the second one provides a sufficient condition under which an equivalence class can be removed from the model;

3. the third one, making use of the second lemma, provides an (exponential) **upper bound** to the **maximum number of equivalence classes** in a minimal model.

The first and the third lemmas together provide an exponential upper bound to the size of a minimal model (**small model theorem**).
**Metric PNL**

Metric PNL (MPNL) is obtained from PNL by adding an infinite set of (pre-interpreted) proposition letters $len_1, \ldots, len_k, \ldots$ for length constraints, that allow one to constrain the length of the current interval to be equal to $1, 2, \ldots$

We prove the decidability of finite satisfiability problem for MPNL~ (or, equivalently, $\text{FO}^2[<, \sim, +1]$) by reducing it to the (decidable) 0-0 reachability problem for vector addition systems (VAS).

**EXPSPACE-hardness** immediately follows from the polynomial-time reduction from the emptiness problem for VAS to the finite satisfiability problem for $\text{FO}^2(\sim, <, +1)$ over data words (two binary relations, that is, the ordering relation $<$ and the equivalence relation $\sim$, and an arbitrary number of unary relations)

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3. $AB \sim_1 \sim_2$
   
   Syntax and Semantics of $AB$
   Previous results
   Undecidability of $AB \sim_1 \sim_2$
The formulas of the logic of Allen’s relations *meets* and *begins*, denoted by $AB$, are recursively defined as follows:

$$\varphi ::= p \mid \neg \varphi \mid \varphi \lor \varphi \mid \langle A \rangle \varphi \mid \langle B \rangle \varphi$$
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SYNTAX AND SEMANTICS OF AB

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\varphi ::= p \mid \neg \varphi \mid \varphi \lor \varphi \mid \langle A \rangle \varphi \mid \langle B \rangle \varphi
$$

$$
\begin{align*}
\langle B \rangle \varphi
\end{align*}
$$

$$
\begin{align*}
\varphi
\end{align*}
$$
Syntax and Semantics of $AB$

The formulas of the logic of Allen’s relations *meets* and *begins*, denoted by $AB$, are recursively defined as follows:

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\varphi ::= p \mid \neg \varphi \mid \varphi \lor \varphi \mid \langle A \rangle \varphi \mid \langle B \rangle \varphi
$$

$AB$ allows one to constrain the length of an interval to be equal to $k$ ($k \in \mathbb{N}$) as well as to constrain an interval to contain exactly one point (endpoints excluded) labeled with a given proposition letter $q$. LTL modalities can be easily expressed in $AB$. 
The satisfiability problem for:

- $AB$ is $\text{EXPSPACE}$-complete on the class of finite linear orders (and on $\mathbb{N}$);

- $AB \sim$ is decidable (but non-primitive recursive hard) on the class of finite linear orders (and undecidable on $\mathbb{N}$).


UNDECIDABILITY OF $AB \sim_1 \sim_2$

We complete the picture by showing that the addition of two (or more) equivalence relations to $AB$ makes the logic undecidable.

**Theorem**

The satisfiability problem for $AB \sim_1 \sim_2$ on the class of finite linear orders is undecidable.

The proof relies on a reduction from the (undecidable) 0-0 reachability problem for counter machines (with two counters) to the satisfiability problem for $AB \sim_1 \sim_2$ on finite linear orders.
**RELATED WORK - 1**

NEXPTIME-completeness of $\text{FO}^2[\prec]$.


NEXPTIME-completeness of $\text{FO}^2[\sim]$.


2-NEXPTIME-completeness of $\text{FO}^2[\sim_1, \sim_2]$.

Undecidability of $\text{FO}^2[\sim_1, \sim_2, \sim_3]$. 


NEXPTIME-completeness of $\text{FO}^2(<, \sim)$ and decidability of $\text{FO}^2(<, \sim, +1)$ on data words (both results have been provided for both finite linear orders and $\mathbb{N}$). 

## RESULTS AND OPEN PROBLEMS

<table>
<thead>
<tr>
<th>Logic</th>
<th>Complexity (on finite linear orders)</th>
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<tbody>
<tr>
<td>PNL (FO²[&lt;])</td>
<td>NEXPTIME-complete - APAL 2009</td>
</tr>
<tr>
<td>PNL ~ (FO²[&lt;, ~])</td>
<td>NEXPTIME-complete - TIME 2014</td>
</tr>
<tr>
<td>PNL ~₁₂ (FO²[&lt;, ~₁, ~₂])</td>
<td>?</td>
</tr>
<tr>
<td>MPNL ~ (FO²[&lt;, ~, +1])</td>
<td>decidable (VASS-reachability) - TIME 2014</td>
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<tr>
<td>AB</td>
<td>EXPSPACE-complete - STACS 2010</td>
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<tr>
<td>AB ~</td>
<td>non-primitive recursive hard - LICS 2013</td>
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<tr>
<td>AB ~₁₂</td>
<td>undecidable - ICTCS 2014</td>
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In addition, we would like to complete the picture for the case of \( \mathbb{N} \) (we know that PNL is NEXPTIME-complete, AB is EXPSPACE-complete, and AB ~ is undecidable over \( \mathbb{N} \)).