An intriguing tiling problem

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joint work with Achim Blumensath and Olivier Carton

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Problem
Given two \textit{regular} languages $K,L$, decide if:

for all $n \in \mathbb{N}$, there exists a \textit{rectangle} of height $n$ with
all \textit{Kolumns} in $K$ and all \textit{Lines} in $L$ (called a
\textit{solution}).
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Example

Does there exists a picture of all heights such that all lined and all columns contain exactly one $a$, ($K = L = b^*ab^*$). (Yes!)
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(Yes!)

Remark

This problem is undecidable.

Proof.

One can use for instance the fact that the tiling of the plane by Wang tiles is undecidable.
Lossy tiling problem

Lossy tiling system
Given two regular languages $K,L$, such that $K$ is closed under letter removal (i.e., subword), decide if:

\[
\text{for all } n \in \mathbb{N}, \text{ there exists a rectangle of height } n \text{ with all } K\text{olumns in } K \text{ and all } L\text{ines in } L.
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Lossy tiling problem

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Remark

Removing lines from a solution yields another solution.

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Given two regular languages $K, L$, such that $K$ is closed under letter removal (i.e., subword), decide if:

for all $n \in \mathbb{N}$, there exists a rectangle of height $n$ with all Kolumns in $K$ and all Lines in $L$.

Remark
Removing lines from a solution yields another solution.

Corollary
It is equivalent to ask for solutions of arbitrary large heights.
A first example

Every line contains exactly one \( a \), and every column at most one \( a \):

\[
L = b^* ab^* \\
K = b^* a^* b^*
\]

There are solutions of arbitrary large height.

Every solution of height \( n \) has width at least \( n \).
A second example

Every line contains exactly one \( a \), \( b \)'s before, and \( c \)'s after:

\[
L = b^*ac^*
\]

Every column is of the same form (\( a \) optional):

\[
K = b^*a^?c^*
\]

There are solutions of arbitrary large height.

Every solution of height \( n \) has width at least \( n \).
A third example

The alphabet is \{a, b, c, d\}.
Every line contains exactly one \(c\) between any two \(a\) or \(b\), and contains exactly one \(b\):

\[
L = ((d^*cd^*)(a+b))^* \cap (b+c+d)^*a(b+c+d)^*
\]

Every column contains either only \(d\)'s and at most one \(c\), or only \(b\)'s and at most one \(a\).

\[
K = b^*a?b^* + d^*c?d^*
\]

There are solutions of arbitrary large height.
Every solution of height \(n\) has width at least \(n(n+1)\).
And?

What is the decidability status for lossy tiling systems?

One sub case known decidable: when $K$ is closed under permutations. (unpublished with A. Blumensath and P. Parys)

Why do we think that it is decidable?
▶ Because no known technique for undecidability seem to apply.
▶ Because in all cases of a positive answer, there is an extremely simple and geometric solution.

Why do we care?
▶ because it is a very simple problem, and it is annoying to not know its answer.
▶ because it is a first step toward solving and understanding logics like $\text{MSO}^+ U$.
And?

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OPEN

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▶ because it is a first step toward solving and understanding logics like MSO+$\cup$. 
**MSO+U and AMSO**

MSO+U is the extension of MSO with the construct

$$\text{U}X \varphi(X)$$

stands for

$$\forall n \exists X \mid X \mid > n \land \varphi(X)$$

It expresses properties over (among others) **infinite words.**
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stands for  
\[ \forall n \exists X \mid X \mid > n \land \varphi(X) \]

It expresses properties over (among others) *infinite words*.

**AMSO** is a logic designed for keeping the number quantifiers (here over \( n \)) while removing the use of set cardinals. . . It expresses properties over (among others) *infinite series of numbers*.

- a series is bounded
- the series tends to infinity
- there are infinitely many values visited infinitely many times
- etc...
As far as decidability of satisfaction is concerned
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\[ \text{AMSO} = \text{MSO} + \bigcup \]

\[ \text{WAMSO} \text{(weak fragment)} \bigcup \text{WAMSO} (2) \]

\( \text{WMSO with one prenex alternation of number quantifiers} = \text{lossy tiling systems} \bigcup \text{MSO} = \text{WMSO} \)

As far as descriptive complexity is concerned

\[ \{ \text{All levels of the projective hierarchy} \}\]

\[ \{ \text{All finite levels of the Borel hierarchy} \}\]

\[ \{ \text{Boolean combinations of the third Borel level} \}\]

\[ \{ \text{Boolean combinations of the second Borel level} \}\]
As far as decidability of satisfaction is concerned

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(WMSO with one prenex
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= lossy tiling systems
As far as decidability of satisfaction is concerned

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{\text{All levels of the projective hierarchy} \{ \text{Hummel} \& \text{Skrzypczak} \}}
{\text{All finite levels of the Borel hierarchy} \begin{array}{l}
\text{\\} \\
\text{\\} \\
\text{\\}
\end{array}}
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\bigcup WAMSO^{(2)} \quad (\text{WMSO with one prenex alternation of number quantifiers}) \\
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\[ \text{AMSO} = \text{MSO} + \bigcup \left( \text{WAMSO} \text{ (weak fragment)} \right) \]
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