Pushdown Vector Addition Systems With States

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Vector Addition Systems With States

VASS
\[ \cong \]
VAS
\[ \cong \]
Petri net

\[
\begin{align*}
&\begin{array}{c}
\begin{array}{c}
5 \\
0
\end{array}
\end{array} \\
&\begin{array}{c}
\begin{array}{c}
x \\
y
\end{array}
\end{array}
\end{align*}
\]

\[
\begin{array}{c}
p \\
q \\
r
\end{array}
\]

\[
\begin{array}{c}
-1, +1 \\
0, -1 \\
0, +2
\end{array}
\]
Pushdown Vector Addition Systems

\[ G \xrightarrow{\text{push}(G)} q \xrightarrow{\text{pop}(G)} r \]

\[ G \xrightarrow{\text{push}(F)} r \xrightarrow{\text{pop}(F)} F \]
Pushdown Vector Addition Systems With States

\[ \begin{align*}
\text{push}(G) & : -1, +1 \\
\text{push}(F) & : 0, -1 \quad \text{and} \quad 0, +2 \\
\text{pop}(G) & : 0, +1
\end{align*} \]
Pushdown VASS in a Nutshell

Richer model for the verification of concurrent systems
- Multi-threaded recursive programs
- One recursive server + unboundedly many finite-state clients
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Study the decidability frontier

VASS + zero-tests  Multi-PDS

VASS  PDS
Reachability for pushdown VASS is Tower-hard [Lazic 12]
- Decidability is open

Reachability is decidable for sub-classes of pushdown VASS
- VASS with one zero-tested counter [Reinhardt 08]
- VASS \cap CFL of finite index [Atig\&Ganty 11]

Boundedness and place-boundedness are decidable for VASS with one zero-tested counter [Bonnet\&Finkel\&Leroux\&Zeitoun 12]

We show that boundedness and termination are decidable for pushdown VASS [Leroux\&Praveen\&Sutre 2014]
- Reachability tree: similar to the VASS case
- Truncation technique adapted to pushdown VASS
The reduced reachability tree is the prefix of the reachability tree obtained as follows

\[ q_{\text{init}}, v_{\text{init}}, \sigma_{\text{init}} \]

\[ q, v, \sigma \]

\[ \sigma \text{ untouched} \]

\[ q, v', \sigma' \]

\[ v \leq v' \]
Finiteness of the Reduced Reachability Tree

Theorem (Leroux&Praveen&Sutre 2014)

Reduced reachability trees are finite.

Proof. By contradiction, assume that it is infinite. The tree is finitely branching. So, by König’s Lemma, there is an infinite branch

\[ q_{\text{init}}, v_{\text{init}}, \sigma_{\text{init}} \rightarrow q_1, v_1, \sigma_1 \rightarrow q_2, v_2, \sigma_2 \cdots \]
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\[ \ldots \]

\[ v \quad v' \geq v \]
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Cardinal of The Reduced Reachability Tree

Very very big...
Very very big…

**Theorem (Leroux & Praveen & Sutre 2014)**

The reduced reachability tree of a pushdown VASS is hyper-Ackermannian, and this bound is tight.
## Open Problem

<table>
<thead>
<tr>
<th></th>
<th>VASS</th>
<th>Pushdown VASS</th>
<th>PDS</th>
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</thead>
<tbody>
<tr>
<td>Cardinal of finite reachability sets</td>
<td>Ack</td>
<td>Hyper-Ack</td>
<td>Exp</td>
</tr>
<tr>
<td></td>
<td>[McAloon’84, Mayr&amp;Meyer’81]</td>
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<tr>
<td>Boundedness Termination</td>
<td>Expspace-c</td>
<td>Tower-h ∩ Hyper-Ack-e</td>
<td>PTime</td>
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<td></td>
<td>[Lipton’76,Rackoff’78]</td>
<td>[Lazic’12]</td>
<td>[Bouajjani&amp;Esparza&amp;Maler’97]</td>
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<tr>
<td>Witnesses</td>
<td>2-Exp</td>
<td>≤ Hyper-Ack</td>
<td>Exp</td>
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<td></td>
<td>[Rackoff’78]</td>
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Minimal size of the witnesses?