Böhm Trees as Higher-Order Recursion Schemes

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Higher-order recursion schemes (HORS)

- Functional programs in abstract form
- Typed grammars generating possibly infinite trees

Example

**Terminals**

- $a : o \rightarrow o$
- $b : o \rightarrow o \rightarrow o$
- $c : o$

**Nonterminals**

- $S : o$
- $F : (o \rightarrow o) \rightarrow o \rightarrow o$
- $G : (o \rightarrow o) \rightarrow o$

**Rules**

- $S = G \ a$
- $F \ f \ x = f \ (f \ x)$
- $G \ f = b \ (f \ c) \ (G \ (F \ f))$

**Starting symbol**

- $S$
Example

\[
S = G \ a
\]
\[
F \ f \ x = f \ (f \ x)
\]
\[
G \ f = b \ (f \ c) \ (G \ (F \ f))
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Example

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\begin{align*}
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Example (Tree Generation)

```
  a  c
 b
  b   a a a c
  b
   a a a a c
    ...
```

Theorem (Ong)

*Monadic Second-Order logic (MSO) is decidable on trees generated by HORS.*
Higher-order recursion schemes

Most existing research

- Higher-order dynamics
- First-order outcome

New questions

- Are HORS any good for representing higher-order entities, i.e. syntax trees with binding?
- Are HORS powerful enough to account for arbitrary (not necessarily closed) functional terms, i.e. higher-order components?
- How do we handle binding-related details?
Alternative presentation of HORS : a fragment of $\lambda Y$-calculus

**Types.** Simple types over one atom $o$.

$$\theta, \theta' ::= o \mid \theta \to \theta'$$

**Terms.**

$$M, N ::= x \mid \lambda x^\theta.M \mid M \cdot N \mid Y_\theta$$

**Typing rules.**

$$\Gamma, x : \theta \vdash x : \theta$$

$$\Gamma \vdash \lambda x^\theta.M : \theta \to \theta'$$

$$\Gamma, x : \theta \vdash M : \theta'$$

$$\Gamma \vdash M \cdot N : \theta'$$

**Reduction.**

$$\lambda x^\theta.M) N \rightarrow_\beta M[N/x]$$

$$Y_\theta M \rightarrow_\delta M (Y_\theta M)$$

$$M \rightarrow_\eta \lambda x^\theta.M x$$

(in the last, $x \notin \text{fv}(M)$ and $M$ has type $\theta \to \theta'$)
Relationship of HORS and \(\lambda Y\)

Example

\[
\begin{align*}
S &= G \ a \\
F \ f \ x &= f \ (f \ x) \\
G \ f &= b \ (f \ c) \ (G \ (F \ f))
\end{align*}
\]

The term on the rhs is a \(\lambda Y\)-term of type \(o\) in context

\[
a : o \rightarrow o \quad b : o \rightarrow o \rightarrow o \quad c : o.
\]

We write \(\Gamma_{\leq 1}\) for contexts with types of \underline{order} at most 1.

Proposition (Salvati, Walukiewicz)

There is a correspondence between HORS and \(\lambda Y\)-terms of the form

\[
\Gamma_{\leq 1} \vdash M : o.
\]
Böhm trees (rather than trees)

Consider the term

$$g : o \to o \to o \vdash \lambda f^{(o \to o) \to o}. Y o (\lambda y^o.f (\lambda x^o.g x y)) : ((o \to o) \to o) \to o.$$ 

Its Böhm tree starts with

$$\lambda f.f(\lambda x_1.g x_1 f(\lambda x_2.g x_2 \cdots)).$$
Result

Question

Can we relate HORS and arbitrary Böhm trees?

\[ \Gamma \vdash M : o \quad \subseteq \quad \Gamma \vdash M : \theta \]

Theorem (FSTTCS’13)

For any \( \lambda Y \)-term \( \Gamma \vdash M : \theta \) there is a term

\[ \Gamma_{rep} \vdash M_{rep} : o \]

with

\[ \Gamma_{rep} = \{ z : o, \ succ : o \rightarrow o, \ var : o \rightarrow o, \ app : o \rightarrow o \rightarrow o, \ lam : o \rightarrow o \rightarrow o \} \]

such that \( M_{rep} \) evaluates to a representation of \( M \)'s Böhm tree, where binders are represented by De Bruijn levels.
De Bruijn levels

**Definition**

**De Bruijn levels** are a variable-naming convention where

- variables are natural numbers,
- each variable is given the smallest index not yet in use.

**Example**

The term

\[ g : o \to o \to o \vdash \lambda f. f \ (\lambda x.g \ x \ (f \ (\lambda y.g \ y \ (f \ y))) \]

can be represented by

\[ 0 : o \to o \to o \vdash \lambda 1.1 \ (\lambda 2.0 \ 2 \ (1 \ (\lambda 3.0 \ 3 \ (1 \ 3)))) \]

**Proposition**

*Two terms* \( M \) *and* \( M' \) *have the same De Bruijn levels representation iff they are* \( \alpha \)-*equivalent.*

(not to be confused with **De Bruijn indices**
Representation of De Bruijn levels in $\lambda Y$

We represent terms with binders as Böhm trees of type $o$ in the context

$$\Gamma_{rep} = \{ z : o, \ succ : o \rightarrow o, \ var : o \rightarrow o, \ app : o \rightarrow o \rightarrow o, \ lam : o \rightarrow o \rightarrow o \}$$

$$\overline{n} = succ \ (succ \ \ldots \ (succ \ z) \ \ldots)$$
Theorem

Let $\Gamma \vdash M : \theta$ be a $\lambda Y$-term.

There exists a $\lambda Y$-term $\Gamma_{rep} \vdash M_{rep} : o$ (a HORS) such that

$$BT(M_{rep}) = \text{rep}(BT(M)).$$

Write $\theta^*$ for $\theta[o \rightarrow o/o]$ and $M^*$ for $M[o \rightarrow o/o]$. There exists a $\lambda$-term

$$\Gamma_{rep} \vdash \downarrow_\theta : \theta^* \rightarrow o \rightarrow o$$

such that, for $\vdash M : \theta$, setting

$$M_{rep} = \downarrow_\theta M^* \ 0$$

validates the equation above.

Proof.

Normalization by evaluation (Filinski & Dybjer) internalized in the $\lambda Y$-calculus.
Extension to $\text{PCF}_f$

**Definition**

The types and terms of $\text{PCF}_f$ are defined as follows.

\[ \theta, \theta' ::= B \mid \theta \to \theta' \]
\[ M, N ::= x \mid \lambda x^\theta . M \mid M \ N \mid Y_\theta \]
\[ tt \mid ff \mid \text{if } M \text{ then } N \text{ else } N' \]

equipped with the standard operational semantics.

**Definition (PCF Böhm trees)**

The notion of (infinite) normal forms

\[
\begin{align*}
\Gamma \vdash \bot : B \\
\Gamma \vdash tt : B \\
\Gamma \vdash ff : B \\
\Gamma, x_1 : A_1, \ldots, x_n : A_n \vdash M : B
\end{align*}
\]
\[
\begin{align*}
\Gamma \vdash \lambda x^\theta . M : \overrightarrow{A} \to B \\
\Gamma \vdash M_i : \theta_i \quad (1 \leq i \leq n) \\
\Gamma \vdash N_1 : B \\
\Gamma \vdash N_2 : B \\
(\overrightarrow{\theta} \to B) \in \Gamma
\end{align*}
\]
\[
\Gamma \vdash \text{if } x \overrightarrow{M} \text{ then } N_1 \text{ else } N_2 : B
\]

They correspond to innocent strategies.
**Consequences**

**Corollary**

*The following problems are recursively equivalent.*

1. *Equivalence of HORS*
2. *Böhm tree equivalence for λY*
3. *Böhm tree equivalence for PCFᵢ*
4. *Contextual equivalence for PCFᵢ with respect to contexts with state and control operators*

By MSO model-checking on HORS

**Corollary**

*The following problems are decidable for PCFᵢ and λY terms:*

1. *Normalizability*
2. *Finiteness*
3. *Finite prefix*

...  

However, MSO on Böhm trees (with a binding predicate) is not decidable.
Thank you!