Half-Positional Two-Player Stochastic Games

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An infinite play:
\[ s_0 a_0 s_1 a_1 \cdots \]

Payoff function:
\[ f: \text{infinite play} \rightarrow \mathbb{R} \]
Borel-measurable and bounded

Player 1 prefers: infinite plays with larger payoff,
Player 2 prefers: infinite plays with smaller payoff.
Two players
stochastic
zero-sum
perfect-information
infinite duration
Strategies:

\[ \sigma : (SA)^* S_1 \rightarrow \Delta(A) \]
\[ \tau : (SA)^* S_2 \rightarrow \Delta(A) \]

Question:

When can we play optimally with a positional (deterministic and memoryless)

\[ \sigma : S_1 \rightarrow A \]
Theorem

If the payoff function is shift-invariant and submixing then the game is half-positional.

Half-positional means: Player 1 has an optimal positional strategy
Definition
A payoff function \( f \) is **shift invariant** if for all infinite plays \( p = p_1p_2 \) where \( p_1 \) is a finite prefix:

\[
f(p_1p_2) = f(p_2)
\]

Definition
A payoff function \( f \) is **submixing** if for all infinite plays \( p \) and all factorizations \( p = u_1v_1u_2v_2\ldots \),

\[
f(p) \leq \max\{f(u_1u_2\ldots), f(v_1v_2\ldots)\}.
\]
Sufficient condition to guarantee the existence of a positional optimal strategy for Player 1:

**Theorem (Kopczynski 06, Gimbert Zielonka 06)**

*Two player deterministic games with submixing and shift-invariant payoff functions.*

**Theorem (Gimbert 07)**

*One player stochastic games with submixing and shift-invariant payoff functions.*

Our main result:

**Theorem**

*Two-player stochastic games with submixing and shift-invariant payoff functions.*
Payoff functions that are shift-invariant and submixing:

- parity
- mean-payoff
- generalized mean-payoff
- limsup
- (discount)
Proof by induction on the number of actions.

Main technical point:

**Lemma**

*For all* $\epsilon > 0$, *in games with shift-invariant payoff functions, both players have* $\epsilon$*-subgame perfect strategies.*

i.e strategies that are not only $\epsilon$-optimal when the game starts but also whatever finite play has already been played.*
Future work

- A necessary condition,
- Infinite but compact action space.