First Cycle Games
or
Why games that are memoryless determined are typically easy to identify

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Highlights 2014
First Cycle Game: An Example

- Two players move a token along edges of a graph.
- Player ● wins if the first cycle has even length; otherwise Player ◆ wins.
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Graph Games

Setting
Two player games, perfect information, finite arena, pure strategies, single initial state.

Agenda
Understand which games are memoryless determined — one of the players has a memoryless winning strategy.

Approach
First Cycle Games, inspired by Ehrenfeucht and Mycielski (’76).
First Cycle Game = \langle Arena, Language \rangle

\[ FC\Gamma \langle A, P \rangle \]

- Arena \( A = (V, E, v_0, \mathbb{U}, \lambda) \) where \( \lambda : V \rightarrow \mathbb{U} \).
- Language of finite strings \( P \subseteq \mathbb{U}^* \).
- A play is an infinite path starting in initial state \( v_0 \in V \).
- A play is won by ● if the sequence of labels on the first cycle on the play is in \( P \); otherwise it is won by ◆.
Examples of $P \subseteq U^*$

- $P = \text{even length.}$
- $P = \text{average weight positive (}U = \mathbb{Q})$.  
- $P = \text{largest priority even (}U = \mathbb{Z})$. 

Finitary versions of classic games.
Examples of $P \subseteq U^*$

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Finitary versions of classic games.

**Goal**
Characterise those $P \subseteq U^*$ such that every FCG $\langle A, P \rangle$ is memoryless determined.
Easy-to-check properties of $P$

$P \subseteq U^*$ is
- **shift-closed** if $a \cdot b \in P \implies b \cdot a \in P$,
- **cat-closed** if $a, b \in P \implies a \cdot b \in P$, $(a, b \in U^*)$.

<table>
<thead>
<tr>
<th>$P \subseteq U^*$</th>
<th>shift-closed</th>
<th>cat-closed</th>
<th>$U^* \setminus P$ cat-closed</th>
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<tbody>
<tr>
<td>Largest priority even</td>
<td>✔</td>
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<tr>
<td>Ave weight positive</td>
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<tr>
<td>Even length</td>
<td>✔</td>
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FCGs that are memoryless determined are typically easy to identify

Theorem. If

1. $P$ is shift-closed, and
2. both $P$ and $\neg P$ are cat-closed,

then every first cycle game $\langle A, P \rangle$ is memoryless determined.
What is the connection to usual infinite duration games?

Decomposition of a play into simple cycles
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Third Cycle!
Greedy Games

Game $G = \langle A, W \rangle$ is arena $A$ and winning condition $W \subseteq U^\omega$.

**Definition**

Game $G$ is $P$-**greedy** if for every play $\pi$:

1. labeling of every cycle of $\pi$ is in $P \implies \pi$ is won by $\blackCircle$;
2. labeling of every cycle of $\pi$ is in $\neg P \implies \pi$ is won by $\blackDiamond$.

**Example**

- Parity Games are $P$-greedy where $P = \text{max priority is even}$.
- Mean-Payoff Games are $P$-greedy where $P = \text{avg weight is positive}$. 
Connecting FCG to Greedy Games

Theorem.
If $G = \langle A, W \rangle$ is $P$-greedy then a memoryless strategy is winning in $G$ iff it is winning in FCG $\langle A, P \rangle$.

Corollary
The following games are memoryless determined (finite arenas):
1. Parity games
2. Mean payoff games
3. Energy games (initial credit problem)
Recipe for proving $G$ is memoryless determined

1. **Finitise** the winning condition of $G$ to get a language $P \subseteq \mathbb{U}^*$.  
2. Show that $G$ is $P\text{-greedy}$.  
3. Show that $P$ is shift-closed, cat-closed, and $\neg P$ is cat-closed.