GAME OR NOT GAME?

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Index / Wadge problem for a class C

INPUT: a language L in C

OUTPUT:
- the minimal (non-det / alternating) index needed to recognize L
- the Wadge degree of L
Game Automata

- a finite alphabet $\Sigma$,
- a finite set of states $Q$,
- an initial state $q_I \in Q$,
- a transition function $\delta : Q \times \Sigma \rightarrow \{(0, q_0) \lor (1, q_1), (0, q_0) \land (1, q_1)\}$,
- a rank function $\text{rank} : Q \rightarrow \mathbb{N}$
\(W_{(1,3)}\)

\[
\begin{align*}
(\Box, m) & \mapsto \langle m \rangle \\
(\square, m) & \mapsto [m]
\end{align*}
\]

\[
\begin{array}{c}
\langle 2 \rangle \\
\langle 3 \rangle \\
[2] \\
\langle 2 \rangle \\
\langle 1 \rangle \\
[3]
\end{array}
\]

\[t \in T_\Sigma, \text{ where } \Sigma = \{\Diamond, \Box\} \times \{1, 2, 3\}\]
Non-deterministic / Alternating

Game

Deterministic

$W_{(1,3)}$
Non-deterministic / Alternating

\[ M = \{ t \in T_{\{a,b\}} : t(0) = t(1) \} \]
Proposition (Duparc, F., M., I I): The class of game languages is the largest class of regular languages:

- extending the deterministic one,

- closed under complementation and substitution,

- and for which substitution preserves the equivalence relations of having the same index and having the same Wadge degree.
Index problem for a class C

INPUT: a language \( L \) in \( C \)

OUTPUT: the minimal (non-det / alternating) index needed to recognize \( L \)
Index problem for a class C

INPUT: a language $L$ in $C$

OUTPUT: the minimal (non-det / alternating) index needed to recognize $L$

Theorem (FMS, LICS 13): The non-deterministic and alternating index problems are decidable for game automata
Index problem for a class $\mathcal{C}$

**INPUT:** a language $L$ in $\mathcal{C}$

**OUTPUT:** the minimal (non-det / alternating) index needed to recognize $L$
Theorem (Niwinski-Walukiewicz 03): Given a regular language \( L \), it is decidable whether \( L \) is recognizable by a deterministic automaton
Theorem (Niwinski-Walukiewicz 03): Given a regular language $L$, it is decidable whether $L$ is recognizable by a deterministic automaton.

Theorem (FMS, LICS 13): Given a regular language $L$, it is decidable whether $L$ is recognizable by a game automaton.
Proof idea: Use of some basic tools from the composition methods.

\[ w \in (\{0, 1\} \cup \Sigma)^* \Sigma \]
Proof idea: Use of some basic tools from the composition methods.

\[ w \in (\{0, 1\} \cup \Sigma)^+ \{0, 1\} \]
Proof idea: Given a regular language $L$, trace unlabelled profile realisation $t$. 
binary profiles

• $Z_0 \times Tr_\Sigma$
• $Tr_\Sigma \times Z_1$
• $Z_0 \times Tr_\Sigma \cup Tr_\Sigma \times Z_1$
• $Z_0 \times Z_1$

unary profiles

• $Z$

non trivial

(labelled traces) (unlabelled traces)
Definition:

- A trace $w$ has non-trivial profile $Z$ in a regular language $M$, if for each realisation $t$ of $w$ either $t^{-1}M$ is trivial or $t^{-1}M = Z$, and for some realisation $t_0$, $t_0^{-1}M = Z$. 
Given $L$, realisation $t$
Given $L$, every trace has at most one profile in a regular language.
\[ M = \{ t \in T_{\{a,b\}} : t(0) = t(1) \} \]

0 has no profile in \( M \)
A regular language is **locally game** if every trace has a profile in it.
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L is game

\[ \downarrow \]

L is locally game
A regular language is **locally game** if every trace has a profile in it.

L is game

\[\uparrow \]

??

L is locally game
Counter-example:

\[ a\text{-reachable} \]

\[ Thin := \{ t \in Tr_{\{a,b\}} \mid ||\{ x \in \text{dom}(t) \mid x \text{ is } a\text{-reachable}\}|| \leq \aleph_0 \} \]
$M, t, \pi$
$M, t, \pi$
$M, t, \pi$
$M, t, \pi$

$t \in Z_0$

$Z_0 \times Z_1$
$M, t, \pi$

$Z_0 \times Tr \cup Tr \times Z_1$

$t \notin Z_0$
$M, t, \pi$

$t$ resolves $M$ up to $\pi$

if there is such a $t$, $\pi$ is $M$-correct
$t$ resolves $M$ up to $\pi$

if there is such a $t$, $\pi$ is $M$-correct regular property
DFA

being locally game

Deterministic parity aut.

being M-correct
$G_M$

$(p, q)$
determine profile (transition and «local» acceptance)

\[ G_M \]
determine priority («global» acceptance)

\( (p, q) \)
Theorem: A regular language $M$ is recognised by a game automaton iff $M$ is locally game and

$$\mathcal{L}(G_M, q_M) = M.$$