On properties of logical sentences with arbitrary monadic predicates

Nathanaël Fijalkow and Charles Paperman

LIAFA
Université Paris Diderot

Highlights of logic, Game and Automata
September, 2013

1The authors are supported by the ANR project FREC, the second author is supported by Fondation CFM.
Introduction

A nice theorem [Barrington, Compton, Straubing and Thérien, 92]:

$$\text{AC}^0 \cap \text{REG} = \text{"almost aperiodic regular languages"}$$
A nice theorem [Barrington, Compton, Straubing and Thérien, 92]:

\[ AC^0 \cap \text{REG} = "\text{almost aperiodic regular languages}" \]

\[ \text{FO}[\mathcal{M}] \cap \text{REG} = \text{FO}[\text{REG}] \]
Introduction

A nice theorem [Barrington, Compton, Straubing and Thérien, 92]:

\[ \text{AC}^0 \cap \text{REG} = "\text{almost aperiodic regular languages"} \]

\[ \text{FO}[\mathcal{N}] \cap \text{REG} = \text{FO}[\mathcal{REG}] \]
Introduction

A nice theorem [Barrington, Compton, Straubing and Thérien, 92]:

$$\mathbf{AC^0} \cap \mathbf{REG} = \text{"almost aperiodic regular languages"}$$

$$\mathbf{FO}[\mathcal{N}] \cap \mathbf{REG} = \mathbf{FO}[\mathcal{REG}]$$

Remarks:

- Some generalisations are long-standing conjectures.
- Proofs rely on deep Circuit Complexity lower bounds.
A nice theorem [Barrington, Compton, Straubing and Thérien, 92]:

\[ \text{AC}^0 \cap \text{REG} = \text{"almost aperiodic regular languages"} \]

\[ \text{FO}[\mathcal{N}] \cap \text{REG} = \text{FO}[\text{REG}] \]

Remarks:

- Some generalisations are long-standing conjectures.
- Proofs rely on deep Circuit Complexity lower bounds.

Aim

Investigate the sub-case of *monadic numerical predicates* over MSO.
Monadic Second Order Logic

\[ \text{MSO}[P] = \exists x \phi(x) \mid \exists X \phi(X) \mid \neg \phi \mid \psi \land \phi \mid x \in X \mid a(x) \mid P(x_1, \ldots, x_r) \]
Monadic Second Order Logic

\[ \text{MSO}[P] = \exists x \phi(x) \mid \exists X \phi(X) \mid -\phi \mid \psi \land \phi \mid x \in X \mid a(x) \mid P(x_1, \ldots, x_r) \]

\(x, x_1, \ldots, x_r\) are positions over a **finite word**
Monadic Second Order Logic

\[
\text{MSO}[P] = \exists x \phi(x) | \exists X \phi(X) | \neg \phi | \psi \land \phi | x \in X | a(x) | P(x_1, \ldots, x_r)
\]

\(x, x_1, \ldots, x_r\) are positions over a finite word
\(X\) is set of positions.
Monadic Second Order Logic

\[ \text{MSO}[P] = \exists x \phi(x) \mid \exists X \phi(X) \mid \neg \phi \mid \psi \land \phi \mid x \in X \mid a(x) \mid P(x_1, \ldots, x_r) \]

\(x, x_1, \ldots, x_r\) are positions over a finite word
\(X\) is set of positions.
\(P\) is a \emph{r-ary numerical predicate}. That is:

\[ P = (P_n) \text{ with } P_n \subseteq \{0, \ldots, n - 1\}^r. \]
Monadic Second Order Logic

$$\text{MSO}[P] = \exists x \phi(x) \mid \exists X \phi(X) \mid \neg \phi \mid \psi \land \phi \mid x \in X \mid a(x) \mid P(x_1, \ldots, x_r)$$

$x, x_1, \ldots, x_r$ are positions over a finite word
$X$ is set of positions.
$P$ is a \textit{r-ary numerical predicate}. That is:

$$P = (P_n) \text{ with } P_n \subseteq \{0, \ldots, n - 1\}^r.$$

$P$ is uniform if there is $Q \subseteq \mathbb{N}^r$ such that

$$P_n = \{0, \ldots, n - 1\}^r \cap Q.$$
Monadic Second Order Logic

$$\text{MSO}[P] = \exists x \phi(x) \mid \exists X \phi(X) \mid \neg \phi \mid \psi \land \phi \mid x \in X \mid a(x) \mid P(x_1, \ldots, x_r)$$

$x, x_1, \ldots, x_r$ are positions over a finite word
$X$ is set of positions.
$P$ is a $r$-ary numerical predicate. That is:

$$P = (P_n) \text{ with } P_n \subseteq \{0, \ldots, n-1\}^r.$$  

$P$ is uniform if there is $Q \subseteq \mathbb{N}^r$ such that

$$P_n = \{0, \ldots, n-1\}^r \cap Q.$$  

Notation:
Class of numerical predicates of arity at most $r$: $\mathcal{N}_r$.
Class of uniform numerical predicates of arity at most $r$: $\mathcal{UN}_r$.  

Monadic Second Order Logic

\[ \text{MSO}[P] = \exists x \phi(x) \mid \exists X \phi(X) \mid \neg \phi \mid \psi \land \phi \mid x \in X \mid a(x) \mid P(x_1, \ldots, x_r) \]

\( x, x_1, \ldots, x_r \) are positions over a finite word
\( X \) is set of positions.
\( P \) is a \textit{r-ary numerical predicate}. That is:

\[ P = (P_n) \text{ with } P_n \subseteq \{0, \ldots, n - 1\}^r. \]

\( P \) is uniform if there is \( Q \subseteq \mathbb{N}^r \) such that

\[ P_n = \{0, \ldots, n - 1\}^r \cap Q. \]

\textbf{Notation:}
Class of all numerical predicates : \( \mathcal{N} \).
Class of all uniform numerical predicates : \( \mathcal{UN} \).
Monadic Second Order Logic

\[ \text{MSO}[P] = \exists x \phi(x) \mid \exists X \phi(X) \mid \neg \phi \mid \psi \land \phi \mid x \in X \mid a(x) \mid P(x_1, \ldots, x_r) \]

\( x, x_1, \ldots, x_r \) are positions over a finite word.
\( X \) is set of positions.
\( P \) is a \textit{r-ary numerical predicate}. That is:

\[ P = (P_n) \text{ with } P_n \subseteq \{0, \ldots, n-1\}^r. \]

\( P \) is uniform if there is \( Q \subseteq \mathbb{N}^r \) such that

\[ P_n = \{0, \ldots, n-1\}^r \cap Q. \]

Examples:

\underline{Arity 2 non uniform predicate}: \( 2x + y = \text{max} \).
\underline{Arity 3 uniform predicate}: \( x + y = z \).
Regular Predicates

A predicate $P$ is regular if, and only if, its a boolean combination of $x \leq y, x \equiv r \mod q, x = y + k$ with $k$ fixed, $\min, \max$.

Notation:
The class of regular predicates: $\mathcal{REG}$.

Examples:
Arity 1 non uniform predicate: $x = \max - 3$.
Arity 2 uniform predicate: $x < y + 3$. 
Straubing and Crane-Beach Properties

$L \in N_e L$ if there is $e$ such that for all words $u, v$:

$$uev \in L \iff uv \in L.$$
Straubing and Crane-Beach Properties

$L \in N_e \mathcal{L}$ if there is $e$ such that for all words $u, v$:

$$uev \in L \iff uv \in L.$$ 

Let $\mathcal{F}[\langle]$ be your favourite fragment of logic and $\mathcal{P}$ a class of numerical predicates.
Straubing and Crane-Beach Properties

\[ L \in \mathcal{N}_e \mathcal{L} \text{ if there is } e \text{ such that for all words } u, v: \]
\[ uev \in L \iff uv \in L. \]

Straubing Property

\[ \mathcal{F}[<, \mathcal{P}] \cap \text{REG} = \mathcal{F}[<, \mathcal{P} \cap \text{REG}] \]

Crane-Beach Property

\[ \mathcal{F}[<, \mathcal{P}] \cap \mathcal{N}_e \mathcal{L} = \mathcal{F}[<] \cap \mathcal{N}_e \mathcal{L} \]
Straubing and Crane-Beach Properties

$L \in \mathcal{N}_e \mathcal{L}$ if there is $e$ such that for all words $u, v$:

$$uev \in L \iff uv \in L.$$ 

 Straubing Property

$$\mathcal{F}[<, \mathcal{P}] \cap \text{REG} = \mathcal{F}[<, \mathcal{P} \cap \text{REG}]$$

- with: $\text{FO}[\land]$ 

 Crane-Beach Property

$$\mathcal{F}[<, \mathcal{P}] \cap \mathcal{N}_e \mathcal{L} = \mathcal{F}[<] \cap \mathcal{N}_e \mathcal{L}$$

- with: $\text{FO}[-], \text{FO}[\leq, \mathcal{N}_1]$
Straubing and Crane-Beach Properties

\[ L \in N_e L \text{ if there is } e \text{ such that for all words } u, v: \]

\[ uev \in L \iff uv \in L. \]

**Straubing Property**

\[ F[\langle, P]\cap REG = F[\langle, P \cap REG] \]

- with: \( FO[\land] \)
- without: ?

**Crane-Beach Property**

\[ F[\langle, P]\cap N_e L = F[\langle] \cap N_e L \]

- with: \( FO[+], FO[\leq, N_1] \)
- without \( FO[+, \times] \)
Crane-Beach for **MSO**

**Theorem**
The languages definable in $\text{MSO}[^<,\mathcal{N}_1]$ are exactly those recognized by *non-uniform advice automata*. 
Theorem
The languages definable in $\text{MSO}[^{<},\mathcal{N}_1]$ are exactly those recognized by non-uniform advice automata.

Corollary
$\text{MSO}[^{<},\mathcal{N}_1]$ has the Crane-Beach Property:

$$\text{MSO}[^{<},\mathcal{N}_1] \cap \mathcal{N}_e\mathcal{L} = \text{MSO}[^{<}] \cap \mathcal{N}_e\mathcal{L}$$
Substitution Property

Definition
A fragment $F[<, P]$ has the Substitution Property if any regular language define by $\varphi(P_1, \ldots, P_k) \in F[<, P]$ can be defined by $\varphi(R_1, \ldots, R_k)$ such that $R_i \in P \cap \mathcal{REG}$ with the same formula $\varphi$. 

Theorem
$\mathcal{MSO}[<, \mathcal{UN}]$ and $\mathcal{MSO}[<, \mathcal{N}]$ have the Substitution Property.

Corollary
1. $F[<, N]$ and $F[<, \mathcal{UN}]$ have the Straubing Property.
2. $F[<, N]$ has almost the Crane-Beach property:
   
   $F[<, N] \cap \mathcal{NE}L = F[<, \mathcal{REG}] \cap \mathcal{NE}L$. 

Definition
A fragment $\mathcal{F}[<, \mathcal{P}]$ has the Substitution Property if any regular language define by $\varphi(P_1, \ldots, P_k) \in \mathcal{F}[<, \mathcal{P}]$ can be defined by $\varphi(R_1, \ldots, R_k)$ such that $R_i \in \mathcal{P} \cap \mathcal{REG}$ with the same formula $\varphi$.

Theorem
$\text{MSO}[<, \mathcal{UN}_1]$ and $\text{MSO}[<, \mathcal{N}_1]$ have the Substitution Property.
Substitution Property

Definition
A fragment $\mathcal{F}[<, \mathcal{P}]$ has the Substitution Property if any regular language defined by $\varphi(P_1, \ldots, P_k) \in \mathcal{F}[<, \mathcal{P}]$ can be defined by $\varphi(R_1, \ldots, R_k)$ such that $R_i \in \mathcal{P} \cap \text{REG}$ with the same formula $\varphi$.

Theorem
\[\text{MSO}[<, \mathcal{UN}_1] \text{ and } \text{MSO}[<, \mathcal{N}_1] \text{ have the Substitution Property.}\]

Corollary
1. $\mathcal{F}[<, \mathcal{N}_1]$ and $\mathcal{F}[<, \mathcal{UN}_1]$ have the Straubing Property.
2. $\mathcal{F}[<, \mathcal{N}_1]$ has \textit{almost} the Crane-Beach property:
   \[\mathcal{F}[<, \mathcal{N}_1] \cap \mathcal{N}_e \mathcal{L} = \mathcal{F}[<, \text{REG}_1] \cap \mathcal{N}_e \mathcal{L}\]