A final coalgebra for the $k$-regular and $k$-automatic sequences

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Introduction

- *k-automatic* and *k-regular* sequences: classes defined by Allouche/Shallit
- A sequence $\sigma \in \mathbb{Z}^\omega$ is *k-automatic* if generated by a deterministic automaton with output in $\{0, \ldots, k - 1\}$...
- ...where $\sigma(n)$ is output after reading $n$ in base $k$. 
Introduction

- \textit{k-automatic} and \textit{k-regular} sequences: classes defined by Allouche/Shallit
- A sequence $\sigma \in \mathbb{Z}^\omega$ is \textit{k-automatic} if generated by a deterministic automaton with output in $\{0, \ldots, k - 1\}$
- \dots where $\sigma(n)$ is output after reading $n$ in base $k$.
- \textit{k-regular} sequences generalize this:

\[
\frac{\text{k-regular}}{\text{k-automatic}} = \frac{\text{weighted automata}}{\text{deterministic automata}}
\]

- This talk: connecting \textit{k-regular} sequences to (abstract) coalgebra and (concrete) behavioural differential equations.
$k$-regular sequences: a definition (for $k = 2$)

We call a sequence (or stream) $\sigma$ 2-regular when there is a finite family of sequences

$$\Sigma = (\sigma_i) \quad i \leq n \in \mathbb{N}$$

with $\sigma_0 = \sigma$, s.t. for all $i \leq n$ the sequences $\text{even}(\sigma_i)$ and $\text{odd}(\sigma_i)$ are linear combinations of sequences from $\Sigma$. Here $\text{even}$ and $\text{odd}$ are defined by

$$\text{even}(\tau)(n) = \tau(2n)$$

and

$$\text{odd}(\tau)(n) = \tau(2n + 1)$$
Derivative and \texttt{zip}

We will reason with the \textit{stream derivative} from the coinductive stream calculus. Definition:

$$\sigma'(n) = \sigma(n + 1)$$

We can define streams and operators \textit{coinductively} by giving the first element and the derivative, e.g.

\[
\begin{align*}
\text{zip}(\sigma, \tau)(0) &= \sigma(0) \\
\text{zip}(\sigma, \tau)' &= \text{zip}(\tau, \sigma')
\end{align*}
\]

gives

\[
\begin{align*}
\text{zip}(\sigma, \tau)(2k) &= \sigma(k) \\
\text{zip}(\sigma, \tau)(2k + 1) &= \tau(k)
\end{align*}
\]

and thus

$$\text{zip}(\text{even}(\sigma), \text{odd}(\sigma)) = \sigma$$
Systems of \texttt{zip}-equations

$k$-regular sequences can be seen as solutions to finite systems of equations.

\[
\begin{align*}
\tau_1 &= \text{zip}(\tau_1^e, \tau_1^o) \\
\vdots \\
\tau_n &= \text{zip}(\tau_n^e, \tau_n^o)
\end{align*}
\]

Example: the sequence of numbers whose base 3 representation does not contain the digit ‘2’

\[
0, 1, 3, 4, 9, 10, 12, 13, 27, 28, 30, 31, \ldots
\]

is a solution to

\[
\begin{align*}
\sigma &= \text{zip}(3\sigma, 3\sigma + \text{ones}) \\
\text{ones} &= \text{zip}(\text{ones}, \text{ones})
\end{align*}
\]

(with \texttt{ones}(0) = 1, \sigma(0) = 0)
Automata as coalgebras

- Automaton (with output in $S$, input in $A$) is coalgebra for the functor $S \times -^A$.
- Semantics $\mathcal{J}[-]$ given by unique morphism into final automaton:

$$
\begin{align*}
X & \xrightarrow{(o, \delta)} S^A^* \\
\exists! \mathcal{J}[-] & \quad (O, \Delta) \\
S \times X^A & \xrightarrow{1_S \times \mathcal{J}[-]} S \times (S^A^*)^A
\end{align*}
$$

Fact: $\mathcal{J}(x)(w) = o(x_w)$
Streams are an instance of this

If $|A| = 1$, note that $S^{A^*} \cong S^\mathbb{N}$ and we get

$$X \xrightarrow{\exists![-]} S^\mathbb{N}$$

$$(o, \delta) \quad (O, \Delta)$$

$$S \times X \rightarrow S \times (S^\mathbb{N})$$

$$O(\sigma) = \sigma(0)$$
$$\Delta(\sigma) = \sigma'$$
Main result (for case $k = 2$)

**Theorem**

A sequence $\sigma$ is 2-regular if and only if it is the unique solution to a system of stream differential equations

$$o(x) = k \quad x' = \text{zip}(x_e, x_o)$$

for a finite set $X$, where $k \in \mathbb{Z}$, and for each $x \in X$, $x_e$ and $x_o$ are given as a linear combination of elements from $X$.

(Also found by Endrullis/Moss/Silva)
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Idea: transform flat systems into guarded systems.

or: move from standard base $k$ numeration to bijective base $k$ numeration.
Construct a system of stream differential equations from the earlier system:

\[
\begin{align*}
\sigma' &= \text{zip}(3\sigma + \text{ones}, 3\sigma') \\
\sigma'' &= \text{zip}(3\sigma', 3\sigma' + \text{ones}') \\
\text{ones}' &= \text{zip}(\text{ones}', \text{ones}) \\
\text{ones}'' &= \text{zip}(\text{ones}', \text{ones}')
\end{align*}
\]

or

\[
\begin{align*}
w' &= \text{zip}(3w + y, 3x) \\
x' &= \text{zip}(3x, 3x + z) \\
y' &= \text{zip}(y, z) \\
z' &= \text{zip}(z, z)
\end{align*}
\]

Add output values to specification and you’re done!
A final coalgebra diagram

Semantics can be given by the following diagram (initiality + finality):

\[
\begin{array}{c}
\xymatrix{
X & S^X \ar[d]^{\eta} \ar[r] & S^N \ar[d]^{(\text{head}, \delta)} \\
S \times (S^X)^{A_2} \ar[u]^{(o, d)} \ar[r] & S \times (S^N)^{A_2}
}
\end{array}
\]

with

\[
\delta(\sigma)(1) = \text{even}(\sigma') \quad \delta(\sigma)(2) = \text{odd}(\sigma')
\]
An isomorphism of final coalgebras

Can be proven using the bijective base $k$ numeration between $\mathbb{N}$ and $(A_k)^*$. 

Gives correspondence with weighted automata (over any semiring $S$).
Application: divide and conquer recurrences

On the Online Encyclopedia of Integer Sequences, some formats for divide and conquer recurrences are given. E.g.

\[
\begin{align*}
a(2n) &= Ca(n) + Ca(n - 1) + P(n) \\
a(2n + 1) &= 2Ca(n) + Q(n)
\end{align*}
\]

where \( P \) and \( Q \) are expressible by a rational g.f.
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Q (asked on oeis.org/somedcgf.html): ‘An open question would be whether all sequences here discussed are 2-regular.’

A: if you replace the condition ‘expressible by a rational g.f.’ by ‘2-regular’ yes (includes all their examples), otherwise no.
Generalizations, conclusions and future work

- Everything told here about 2 works for any $k \geq 2$.
- We established a correspondence between rational power series in $k$ (noncomm.) variables and $k$-regular sequences over arbitrary semirings.
- ...allowing us to translate back and forth between recurrences and systems of stream differential equations.
- Future work: how about $k$-algebraic sequences? . . . further investigate the connections with recurrences.
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