Expressive Completeness of some logics – proof by games

Agnieszka Kułacka

Imperial College London

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Special thanks to Professor Ian Hodkinson for supervising this project and the organisers of Highlights of Logic, Games and Automata for accepting this presentation.
Expressive completeness, used as a name of a field of enquiry, studies the power of some logics with respect to fragments of first-order logic over some class of frames.

A modal logic is said to be expressively complete over some class of frames iff every formula of first-order logic can be expressed by a formula of this modal logic.
1968  Hans Kamp proved that the temporal logic with connectives Until and Since is expressively complete over the class of all Dedekind complete linear flows of time.

1979  Jonathan Stavi introduced additional connectives and proved that this enriched logic is expressively complete over the class of all linear flows of time.
Yde Venema proved that CDT (an interval logic with connectives Chop, D, T) is expressively complete over linear flows of time with respect to 3-variable fragment of first-order logic.

Kousha Etessami, Moshe Vardi and Thomas Wilke showed that a temporal logic with connectives Future, Past, Tomorrow and Yesterday is expressively complete over linear flows of time with respect to 2-variable fragment of first-order logic.
Related work

**2009** Davide Bresolin, Valentin Goranko, Angelo Montanari and Guido Sciavicco showed that an interval temporal logic Non-strict Propositional Neighbourhood Logic ($PNI\pi^+$) is expressively complete over linear flows of time with respect to 2-variable fragment of first-order logic.
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Proof procedure

Duplicator has a winning strategy in an n-pebble game

Duplicator has a winning strategy in a stone game

n-variable FO does not distinguish between two models (playing boards)

A temporal logic does not distinguish between two models (playing boards)

(1) EF theorem for n-pebble games

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4 Contribution

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Expressive Completeness of some logics
n-pebble game

\[ P_k^n(\mathcal{A}, g_0, \mathcal{B}, h_0) \]

There are \( n \) pairs of pebbles \( \rho_i, \pi_i \) (\( 1 \leq i \leq n \)), \( k \) rounds of game, two structures with a finite purely relational signature, \( \mathcal{A}, \mathcal{B} \), and two initial partial assignments \( g_0, h_0 \) of variables \( x_1, x_2, \ldots, x_n \) to elements of each structure.

There are two players Spoiler and Duplicator.

For each round \( t \) we define positions of pebbles by \( (g_t, h_t) \). \( g_t(x_i) = a_i \) indicates that pebble \( \rho_i \) is placed on element \( a_i \) of structure \( \mathcal{A} \). How?
n-pebble game

At start of round $t$ positions of pebbles: $(g_{t-1}, h_{t-1})$.

Spoiler selects $\rho_i$ or $\pi_i$ and places it on a selected element of its structure, Duplicator responds by placing the other pebble on a corresponding element of the other structure. We define the new positions of pebbles by $(g_t, h_t)$.

After $k$ rounds, the game ends. Who wins?
n-pebble game

Let $D_t \subseteq \{x_1, ..., x_n\}$ be the domain of $g_t, h_t$.

Duplicator wins this play of the game iff for every $t$, 
\{(g_t(x_i), h_t(x_i)) : x_i \in D_t\} is a partial isomorphism from $A$ to $B$; i.e. for every atomic formula $\alpha$ written with variables taken from $D_t$, we have

$$A, g_t \models \alpha \iff B, h_t \models \alpha.$$
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4 Contribution
Ehrenfeucht-Fraïssé theorem for n-pebble games

Let $L$ be a finite purely relational signature, let $\mathcal{A}, \mathcal{B}$ be $L$-structures with domains $M, N$, respectively, and let $k, n \geq 0$ be integers. Let $D \subseteq \{x_1, \ldots, x_n\}$. Then for all assignments $g : D \rightarrow M$ and $h : D \rightarrow N$, the following are equivalent:

- Duplicator has a winning strategy in $P^n_k(\mathcal{A}, g, \mathcal{B}, h)$,
- for every $L$-formula $\psi$ of quantifier depth at most $k$ and whose free variables are in $D$ and all variables among $x_1, \ldots, x_n$, we have $\mathcal{A}, g \models \psi \iff \mathcal{B}, h \models \psi$. 

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4 Contribution
There is a pair of stones $\rho, \pi$, $k$ rounds of game, two Kripke models $\mathcal{M}, \mathcal{N}$. At the start of the game the position of stones is $(m_0, n_0) \in \mathcal{M} \times \mathcal{N}$.

There are two players Spoiler and Duplicator.

For each round $t$ we define a position of stones by $(m_t, n_t)$, which corresponds to stone $\rho$ placed on element $m_t$ and stone $\pi$ placed on element $n_t$. How? What are the moves?
Let the current position of the stones be \((m, n)\).

By a forward move in \(M\), we mean selecting \(m^* \in M\) such that \(m < m^*\) and placing a stone \(\rho\) on it.

By a backward move in \(M\), we mean selecting \(m^* \in M\) such that \(m^* < m\) and placing a stone \(\rho\) on it.

Similarly, we define forward and backward moves in \(N\). How to play?
At start of round $t$ the position of the stones: $(m_{t-1}, n_{t-1})$. Spoiler selects either $\rho$ or $\pi$, and then makes a move in a chosen direction by placing a selected stone on a selected element. Duplicator responds by making a move in the same direction and choosing an element of the other structure by placing on it the corresponding stone. We define the new position of stones by $(m_t, n_t)$.

After $k$ rounds, the game ends. Who wins?
Duplicator wins this play of the game iff for every $0 \leq t \leq k$, for every $p \in PROP$, where $PROP$ is a fixed finite set of atoms, we have

$$\mathcal{M}, m_t \models p \iff \mathcal{N}, n_t \models p.$$
FP logic – quick reminder

- Formulas $\phi$ of FP are

  $$\phi ::= p \mid \top \mid \neg \phi \mid \phi \land \phi \mid F\phi \mid P\phi,$$

  where $p \in PROP$.

- Formulas are evaluated at points in a Kripke model $(T, <, h)$, where the accessibility relation $<$ is the earlier-later relation (linear).

- $F\phi$ means $\phi$ is true at some future point of time, $P\phi$ similar for the past.
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4 Contribution
Let $\mathcal{M}, \mathcal{N}$ be Kripke models of linear flows of time. Let $k \geq 0$ be an integer. Then for every $m \in \mathcal{M}$ and $n \in \mathcal{N}$, the following are equivalent:

- Duplicator has a winning strategy in $S_k(\mathcal{M}, m, \mathcal{N}, n)$,
- for every $FP$-formula $\psi$ of temporal operator depth at most $k$ written with propositional atoms $p \in PROP$, we have $\mathcal{M}, m \models \psi \iff \mathcal{N}, n \models \psi$. 

Ehrenfeucht-Fraïssé theorem for stone games
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4 Contribution
Strategy transfer theorem

Let $\mathcal{M}, \mathcal{N}$ be Kripke models of linear flows of time. Let
$\mathcal{A}(\mathcal{M}) = (M, \{<^A\} \cup \{P^A : p \in PROP\})$ and
$\mathcal{B}(\mathcal{N}) = (N, \{<^B\} \cup \{P^B : p \in PROP\})$ be the first-order
$L(\text{PROP})$-structures constructed from $\mathcal{M}, \mathcal{N}$, respectively. Let
$k \geq 0$ be an integer.

Then for all $m \in \mathcal{M}$ and $n \in \mathcal{N}$, the following two are equivalent:

(stone) Duplicator has a winning strategy in $S_k(\mathcal{M}, m, \mathcal{N}, n)$,

(pezbble) Duplicator has a winning strategy in
$P^2_k(\mathcal{A}(\mathcal{M}), \{(x_1, m)\}, \mathcal{B}(\mathcal{N}), \{(x_1, n)\})$. 

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Fragment of a proof (stone) $\Rightarrow$ (pebble)
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Fragment of a proof (stone) $\Rightarrow$ (pebble)

Spoiler’s move in the public game

$m_1 \rightarrow m_2' \rightarrow m_2$
Fragment of a proof (stone) $\Rightarrow$ (pebble)

Spoiler’s move in the public game

Duplicator establishes her response in the private game due to her winning strategy ($s_1$ copied and replaces $s_2$)
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Fragment of a proof (stone) $\Rightarrow$ (pebble)

Spoiler’s move in the public game

\[
\begin{array}{c}
\text{m1} \quad \text{m2'} \quad \text{m2} \\
\end{array}
\quad \begin{array}{c}
\text{n1} \quad \text{n2} \\
\end{array}
\]

Duplicator establishes her response in the private game due to her winning strategy ($s_1$ copied and replaces $s_2$)

\[
\begin{array}{c}
\text{m1} \quad \text{m2'} \\
\end{array}
\quad \begin{array}{c}
\text{n1} \quad \text{n2'} \\
\end{array}
\]

Duplicator’s response in the public game

\[
\begin{array}{c}
\text{m1} \quad \text{m2'} \\
\end{array}
\quad \begin{array}{c}
\text{n1} \quad \text{n2} \quad \text{n2'} \\
\end{array}
\]
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Equivalence of formulas theorem

Let $\mathcal{C}$ be a class of linear flows of time $\mathcal{F} = (T, <)$. Then for every finite set of propositional atoms $PROP$, for every $FO^2$ formula $\psi(x, L(PROP))$, there is a formula $A(PROP)$ of $FP$ logic, whose standard translation $A^x(x, L(PROP))$ is equivalent to $\psi(x, L(PROP))$ over the class of $L(PROP)$-structures constructed from models of linear flows of time in $\mathcal{C}$. 
## Contributions

Proofs by games – uniform approach

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<th>first order logics</th>
<th>all linear flows of time</th>
<th>all Kripke frames</th>
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Thank you. Questions?