The Complexity of Admissibility in $\omega$-Regular Games

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Highlights of Logic, Games and Automata
21st of September 2013
Controller synthesis
Controller synthesis
Controller synthesis

controllers

strategies
Models of rationality

- Nash equilibria $\iff$ no player has interest in deviating.
- Regret minimization $\iff$ players prefer moves that would induce less regret had they known the other players strategy.
- Elimination of dominated strategies $\iff$ players eliminate “bad” strategies

$\Rightarrow$ In all cases it is assumed everybody knows and uses the model of rationality.
Iterative elimination of dominated strategies

- What is a “bad” strategy? \( \sigma \) is strictly dominated by \( \sigma' \) if
  - for all profiles of the other players, if \( \sigma \) wins, so does \( \sigma' \).
  - for some profile of the other players, \( \sigma \) loses while \( \sigma' \) wins.

Each player eliminates its dominated strategies.
Repeat until stabilized.
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Our setting

- Turn based games on graphs.
- Objective of player $i$: $\text{WIN}_i \subseteq V^\omega$. 

Muller objectives: $\rho \in \text{WIN}_i \iff \inf(\rho) \in F$.

$\Rightarrow$ Generalizes B"uchi and parity conditions.

Weak Muller objectives: $\rho \in \text{WIN}_i \iff \occ(\rho) \in F$.

$\Rightarrow$ Generalizes safety and reachability conditions.
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Admissibility

- **Dominance:** $\sigma'_i \succ_S \sigma_i$ if $\sigma'_i$ strictly dominates $\sigma_i$ w.r.t $S^n$.

- **Iterative admissibility:** $S^0_i = S_i$ and

$$S^{n+1}_i := S^n_i \setminus \{\sigma_i \mid \exists \sigma'_i \in S^n_i, \sigma'_i \succ_S \sigma_i\}.$$  

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**Goal**: compute $S^*$ or at least decide properties thereof.
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**Remark**

$S^*$ is well defined and is reached after a finite number of iterations.

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**Decision problems on $S^*$**

The **winning coalition problem**: Given $W, L \subseteq P$, does there exist $\sigma_P \in S^*$ such that all players of $W$ win the game, and all players of $L$ lose.

The **model-checking under admissibility problem**: Given $\varphi$ an LTL formula, is it the case that for any profile $\sigma_P \in S^*$, $\text{Out}(\sigma_P) \models \varphi$?
Values

Introduced in [Berwanger, STACS'07]

- If there is a winning strategy
  \[\Rightarrow\] admissible strategies are the winning ones.
- It is impossible to win
  \[\Rightarrow\] all strategies are admissible.

Remark

A player should never decrease its own value.
The value depends on \(S\).

\[\Rightarrow\] How to compute those values?

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  \[\Rightarrow\] What are the admissible strategies in this case?

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- If there is a winning strategy: value 1.
  - Admissible strategies are the winning ones.
- It is impossible to win: value $-1$.
  - All strategies are admissible.
- Otherwise: it is possible to win, but only with the help of others: value 0.

What are the admissible strategies in this case?

\[ \begin{align*}
V_1 &= 0 \\
V_2 &= 1 \\
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Safety objectives: a local notion of dominance

- **Objective:** avoid *Bad* states
- **Existence of a winning strategy depends only on:**
  - the current state
  - *Bad* states visited
    - unfold the graph to keep this information
    - size: $|V| \times 2^{|P|}$.

- In unfolded safety games the rule to never decrease one’s own value is **sufficient** for admissibility.
- The structure of the unfolding avoid explosion in complexity.

**Theorem**

*The winning coalition problem is PSPACE-complete for safety.*
In general: the local condition is not sufficient

Prefix-independent objectives

In case the value is 0, need to allow other players to help.
"Help!"-state for $i$ where $j \neq i$ has several choices with value $\geq 0$ for $i$, while not changing the value for $j$.

Admissible strategies should be winning if the other players played fairly in those states.

This gives rise to automaton $A_n$ recognizing $Out(S_n)$ with circuit winning condition.

In turn, $A_n$ is used to compute the values at the next step.
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Complexity for Objectives defined by Circuits

Theorem (Winning coalition problem)
- The winning coalition problem PSPACE-complete for circuits.
- The winning coalition problem with Büchi objectives is in NP ∩ coNP.
- The winning coalition problem for weak circuit is PSPACE-complete.

Theorem (Model-checking under admissibility problem)
The model-checking under admissibility problem is PSPACE-complete for games where the winning condition of each player is given by a circuit condition.
Summary

- Automata representing all outcomes of admissible strategies.
- Algorithms with **tight complexity bounds** to compute the set of all outcomes of iteratively admissible strategies.
- Application to model-checking of LTL assuming all players follow rationality.
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- Algorithms with *tight complexity bounds* to compute the set of *all outcomes* of iteratively admissible strategies.
- Application to model-checking of LTL assuming all players follow rationality.

Future work

- Extension to *quantitative* games.
- Implementation.
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Thank you