Highlights of Logic, Games and Automata

Emanual Kieroński, Jakub Michaliszyn, Jan Otop
Modal logic

- Many different modal logics (K4, S5, CTL, LTL, ATL, HS, CTL*K)
- Many applications in verification, planing, linguistics
- Many proofs, many papers
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- Many applications in verification, planing, linguistics
- Many proofs, many papers
- Our area of interest: a comprehensive study on the satisfiability problem.
Kripke semantics

- Kripke structure — a frame + a labelling.
Kripke semantics

- Kripke structure — a *frame* + a *labelling*.
- $\mathcal{K}$-SAT — local satisfiability problem w.r.t. $\mathcal{K}$.
- $\mathcal{K}$-GSAT — global satisfiability problem w.r.t. $\mathcal{K}$.
Kripke semantics

- Kripke structure — a \textit{frame} + a \textit{labelling}.
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- $\mathcal{K}$-GSAT — global satisfiability problem w.r.t. $\mathcal{K}$.

Our ultimate goal

For all first-order definable classes $\mathcal{K}$, determine the decidability and complexity of $\mathcal{K}$-SAT and $\mathcal{K}$-GSAT.

We are also interested in finite satisfiability.
Negative results

- (E. Hemaspaandra, "The Price of Universality", 1996) \( \mathcal{K}\)-GSAT is undecidable for some \( \forall\text{FO}\)-definable \( \mathcal{K} \).

- (E. Hemaspaandra, H. Schnoor, MFCS 2011) \( \mathcal{K}\)-SAT is undecidable for some \( \forall\text{FO}\)-definable \( \mathcal{K} \).

Jakub Michaliszyn (Wrocław) Questions are welcomed! Highlights’13 4 / 12
Negative results


- (E. Hemaspaandra, H. Schnoor, MFCS 2011) $K$-SAT is undecidable for some $\forall$FO-definable $K$.

- (E. Kieroński, J. Michaliszyn, J. Otop, FSTTCS 2011) $K$-GSAT and $K'$-SAT are undecidable for some $\forall$FO$^3$-definable $K$ and $K'$ (holds also for finite satisfiability).

\[
\neg xRy \lor \neg xRz \lor yRz \lor zRy \lor yRx \lor zRx
\]
Positive results

**Standard translation**

Is $\varphi$ satisfied w.r.t. the class defined by $\Phi$? $\rightarrow$ Is $\Phi \land ST(\varphi)$ satisfiable?
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Decidability in many interesting cases (even multimodal), including:

- $\text{FO}^2$:
  - with one transitive relation (W. Szwast, L. Tendera, 2012),
  - with counting quantifiers (I. Pratt-Hartmann, 2005),

- Guarded Fragment:
  - with fixed points (E. Grädel, I. Walukiewicz, 1999),
  - with the transitive closure operator in guards (J. Michaliszyn, 2009).

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- High complexity.

Questions are welcomed!

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Positive results

For any $\mathcal{K}$ definable by universal Horn formulas, $\mathcal{K}$-SAT and $\mathcal{K}$-GSAT are decidable.

J. Michaliszyn, J. Otop, LICS 2012
Positive results

For any $\mathcal{K}$ definable by universal Horn formulas, $\mathcal{K}$-SAT and $\mathcal{K}$-GSAT are decidable.

Also finite satisfiability of modal logic is decidable w.r.t. the classes definable by universal Horn formulas.
General satisfiability

<table>
<thead>
<tr>
<th>Type</th>
<th>$\mathcal{K}_\Phi$-GSAT</th>
<th>$\mathcal{K}_\Phi$-SAT</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1+</td>
<td>EXPTIME-c</td>
<td>PSPACE-c</td>
</tr>
<tr>
<td>S1−</td>
<td>PSPACE-c</td>
<td>NP-c</td>
</tr>
<tr>
<td>S2+</td>
<td>NP-c</td>
<td>PSPACE-c</td>
</tr>
<tr>
<td>S2−</td>
<td>NP-c</td>
<td>NP-c</td>
</tr>
<tr>
<td>S3+</td>
<td>impossible</td>
<td></td>
</tr>
<tr>
<td>S3−</td>
<td>NP-c</td>
<td>NP-c</td>
</tr>
</tbody>
</table>

Except for some trivial formulas like $xRx \land (xRx \Rightarrow \bot)$. 
## Finite satisfiability

<table>
<thead>
<tr>
<th>Type of $\Phi$</th>
<th>$\mathcal{K}_\Phi$-GFINSAT</th>
<th>$\mathcal{K}_\Phi$-FINSAT</th>
</tr>
</thead>
<tbody>
<tr>
<td>S3+, S3–</td>
<td>FMP, NP-c</td>
<td></td>
</tr>
<tr>
<td>S2+, S2–</td>
<td>NEXPTIME</td>
<td></td>
</tr>
<tr>
<td>S1+ &amp; “merges”</td>
<td>Lack of FMP (always!), PSPACE-c</td>
<td>FMP, PSPACE-c</td>
</tr>
<tr>
<td>S1+ &amp; not “merges”</td>
<td>FMP, EXPTIME-c</td>
<td>FMP, PSPACE-c</td>
</tr>
<tr>
<td>S1–</td>
<td>FMP, PSPACE-c</td>
<td>FMP, NP-c</td>
</tr>
</tbody>
</table>
Finite vs. General

J. Michaliszyn, J. Otop, P. Witkowski, Gandalf 2012

- There is an undecidable logic that is finitely decidable
- There is a decidable logic that is finitely undecidable
Transitiveness

- Transitive modalities are popular in practice:
- $F, G$ of LTL
- $B, D, L$ of HS logic
- $K_i, C_G$ of epistemic logic

For any $\mathcal{K}$ of transitive frames definable by universal formulas, $\mathcal{K}$-SAT and $\mathcal{K}$-GSAT are decidable. The same holds for the finite satisfiability problem.

J. Michaliszyn, J. Otop, CSL 2013
So what?

Our ultimate goal for all first-order definable classes $K$, classify $K$-SAT, $K$-GSAT (and their finite counterparts) w.r.t. the decidability status and the complexity.

Why?

Better understanding, easy modifications, unified theory.

The meta-problem: Input: A first-order formula $\Phi$ that defines a class of frames $K$. Question: Is $K$-SAT decidable? Is the meta-problem decidable?

Questions are welcomed!
So what?

Our ultimate goal

For all first-order definable classes $\mathcal{K}$, classify $\mathcal{K}$-SAT, $\mathcal{K}$-GSAT (and their finite counterparts) w.r.t. the decidability status and the complexity.
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Why?
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- Easy modifications
- Unified theory

The “metaproblem”
Input: A first-order formula \( \Phi \) that defines a class of frames \( \mathcal{K} \).
Question: Is \( \mathcal{K}\)-SAT decidable?

Is the metaproblem decidable?
Thank you for your attention!

Summary

- We study the **satisfiability** problem of **modal logic** over **first-order** definable classes of **frames**.
- In some cases the problem is **undecidable**.
- There are wide classes of formulas that lead to **decidable** problems (Horn formulas, transitive formulas, $\text{FO}^2$, $\text{GF}$).
- Our goal: to classify them all.

Open: Is the “metaproblem” decidable?

Input: A first-order formula $\Phi$ that defines a class of frames $\mathcal{K}$.

Question: Is $\mathcal{K}$-SAT decidable?