# Register Games on Infinite Ordered Domains

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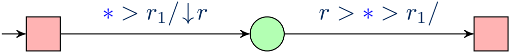
$(\mathbb{N},>)$ or $(\mathbb{Q},>)$

**Register games** extend two-player zero-sum turn-based games to *infinite* alphabets.



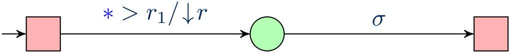

*Play* starts in the *configuration* $(q_0, 0^R)$, with all registers $0$. Adam provides a data. It is compared with the register values. Based on the result, the game dictates a register *assignment* action and a successor state (unique). We update the registers and move into the new configuration. Repeat the same for Eve.

The parity *acceptance* is defined on vertices.

Strategies don't see the data: Adam $\mathsf{Tst}^* \to \mathbb{N}$, Eve $\mathsf{Tst}^+ \to \mathbb{N}$.

Solving such games is **undecidable**: they can simulate 2CMs.

In **one-sided register games**, only Adam provides a data, while Eve gives a label from a *finite* alphabet.

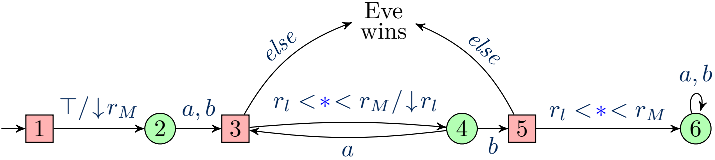



Strategies become: Adam $\Sigma^* \to \mathbb{N}$, Eve $\mathsf{Tst}^+ \to \Sigma$.

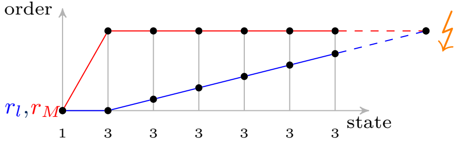
Our results:

- They are **decidable** for $(\mathbb{N}, >)$ (ExpTime in # registers).
- **Finite memory** suffices for Eve.
- Either Eve wins or Adam wins (**determinacy**).

**Example.** Registers $R = \{r_l, r_M\}$, Eve finite alphabet is $\{a, b\}$.




In $\mathbb{N}$, Eve wins by always outputting $a$. Adam loses because:




But Adam wins in the domain $\mathbb{Q}$.

## Reduction to finite-arena games

Treat tests-assignments as labels, modify a winning condition:

$$EveWinPlays = \{\pi \mid \mathsf{tst}_1\mathsf{asgn}_1\ldots \textit{ of } \pi \textit{ is } \boldsymbol{feasible} \Rightarrow \pi \models \alpha\}.$$

In the example, the tests-assignments of the play $1, 2, (3, 4)^\omega$ are not feasible in $\mathbb{N}$ ($\Rightarrow$ winning for Eve).

Sequence $\mathsf{tst}_1\mathsf{asgn}_1\ldots$ is *feasible* $\equiv$ generated by some data values:
$(r_1 < * < r_2)(\downarrow r_1 r_2)(r_1 = r_2 = *)$ is feasible but
$(r_1 < * < r_2)(\downarrow r_1 r_2)(r_1 < * < r_2)$ is not.

Detecting unfeasible sequences in $\mathbb{Q}$ is easy: spot inconsistencies like above.

In $\mathbb{N}$ it is hard. Studying constraint sequences (page 5) gives us *det-max* automaton that can detect unfeasible sequences.

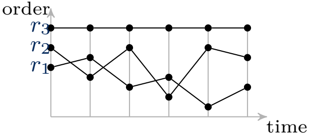
Finally, with a help of a pigeon†, we get rid of det-max automata and get games with standard $\omega$-regular winning conditions.

†: the proof relies on determinacy and finite-mem of $\omega$-regular games, and the pigeon hole

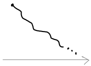
**Satisfiability of constraint sequences in $\mathbb{N}$**

A *constraint* defines the relation between the registers.

A constraint *sequence* describes how these relations evolve:

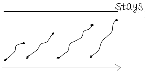



We prove that a constraint sequence is *unfeasible* in $\mathbb{N}$ iff it



has an infinite decreasing *chain*

OR




has unbounded decreasing or increasing *trespassing chains*

We construct a det-max automaton that reads constraint sequences and spots such chains.

Finally, we apply these results to **transducer synthesis**:
 input: deterministic “transducer-like” register automaton $S$
output: transducer $T$ such that $L(T) \subseteq L(S)$, or ‘unrealisable’.
And show it is decidable.

The draft of the paper (raw!) is available at
`https://arxiv.org/pdf/2004.12141.pdf`