Complexity of solving games with combination of objectives using separating automata

Highlights 2020

Ashwani Anand, Chennai Mathematical Institute, India

This is a joint work with Nathanaël Fijalkow and Jérôme Leroux
LaBRI, France

Definitions and notations

We consider games on labelled graphs played between two players, Adam and Eve, with certain winning objectives on the infinite sequences of labels generated by playing.

Some of the popular objectives are:

Definitions and notations

We consider games on labelled graphs played between two players, Adam and Eve, with certain winning objectives on the infinite sequences of labels generated by playing.

Some of the popular objectives are:

- Parity (P): Eve wins the game, if the maximum of the infinitely many times occuring colours is even. Adam wins, otherwise.

Definitions and notations

We consider games on labelled graphs played between two players, Adam and Eve, with certain winning objectives on the infinite sequences of labels generated by playing.

Some of the popular objectives are:

- Parity ($\mathtt{P}$): Eve wins the game, if the maximum of the infinitely many times occuring colours is even. Adam wins, otherwise.
- Mean-Payoff ($\mathtt{MP}$): Eve wins the game, if the average limit of the infinite sequence is non-negative. Adam wins, otherwise.

Definitions and notations

We consider games on labelled graphs played between two players, Adam and Eve, with certain winning objectives on the infinite sequences of labels generated by playing.

Some of the popular objectives are:

- Parity (P): Eve wins the game, if the maximum of the infinitely many times occuring colours is even. Adam wins, otherwise.
- Mean-Payoff (MP): Eve wins the game, if the average limit of the infinite sequence is non-negative. Adam wins, otherwise. We will consider two variants: $\overline{\mathsf{MP}}$, with $\limsup$ of averages, and $\underline{\mathsf{MP}}$, with $\liminf$ of the averages.

Games with combination of objectives

- Games with multi-dimensional labels.

Games with combination of objectives

- Games with multi-dimensional labels.
- Denoted as $W_1 \vee W_2$, in two dimension.

Games with combination of objectives

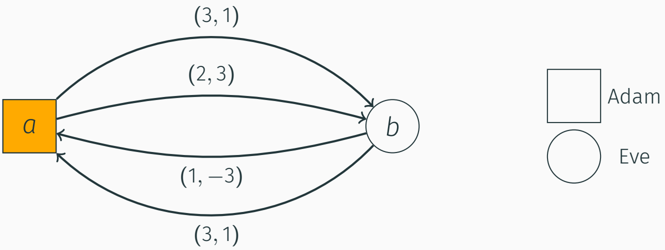
- Games with multi-dimensional labels.
- Denoted as $W_1 \vee W_2$, in two dimension.
- Eve wins $W_1 \vee W_2$, if projection of the infinite sequence on first coordinate satisfies W_1, or that on second coordinate satisfies W_2.

Games with combination of objectives

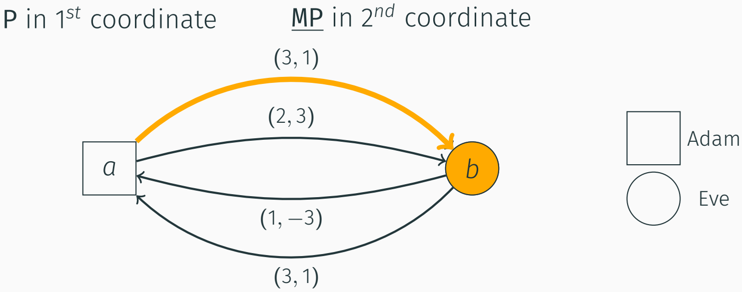
- Games with multi-dimensional labels.
- Denoted as $W_1 \vee W_2$, in two dimension.
- Eve wins $W_1 \vee W_2$, if projection of the infinite sequence on first coordinate satisfies W_1, or that on second coordinate satisfies W_2.
- We give the algorithms for solving the games with combination of objectives by constructing *separating automata* for them, combining those for the individual objectives as black boxes.

Example: $\mathsf{P} \vee \underline{\mathsf{MP}}$

P in 1^{st} coordinate $\underline{\mathsf{MP}}$ in 2^{nd} coordinate



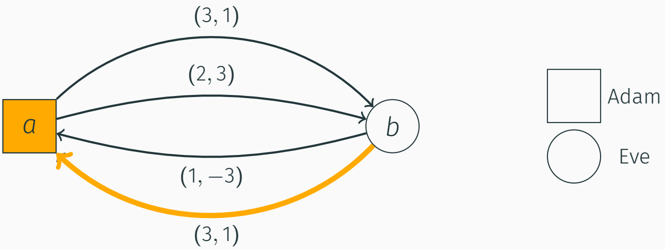
Example: P ∨ MP



$(3, 1)$

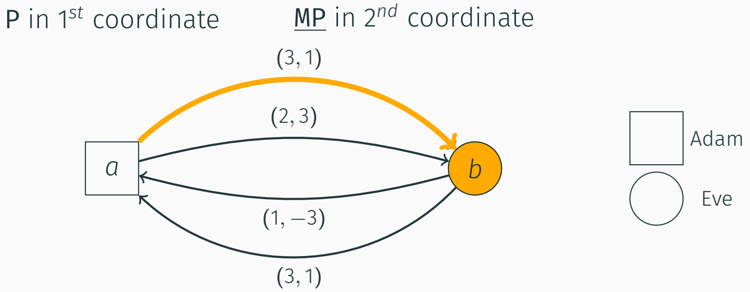
Example: $\mathsf{P} \vee \underline{\mathsf{MP}}$

P in 1^{st} coordinate $\underline{\mathsf{MP}}$ in 2^{nd} coordinate



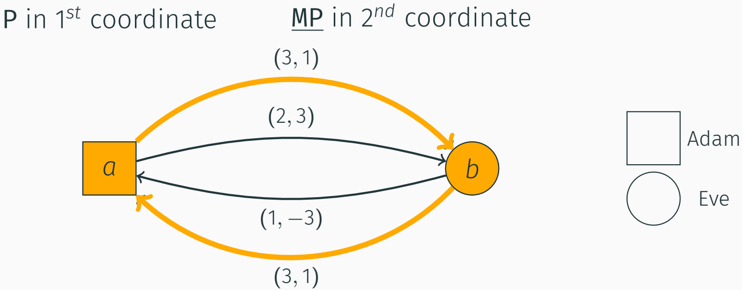
$(3, 1)(3, 1)$

Example: P ∨ $\underline{\text{MP}}$



$(3,1)(3,1)(3,1)$

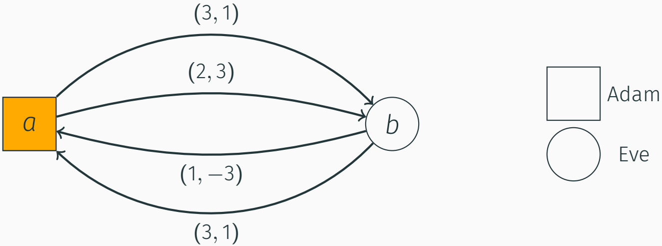
Example: $\mathsf{P} \vee \underline{\mathsf{MP}}$



$((3,1)(3,1)(3,1)(3,1))^{\omega} \models \mathsf{P} \vee \underline{\mathsf{MP}}$

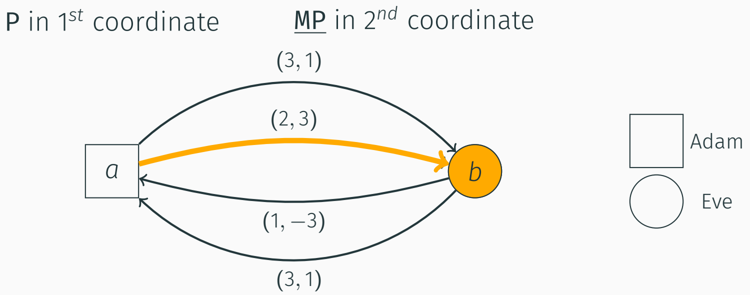
Example: $\mathsf{P} \vee \underline{\mathsf{MP}}$

P in 1^{st} coordinate $\underline{\mathsf{MP}}$ in 2^{nd} coordinate



$((3,1)(3,1)(3,1)(3,1))^{\omega} \models \mathsf{P} \vee \underline{\mathsf{MP}}$

Example: $\mathsf{P} \vee \underline{\mathsf{MP}}$

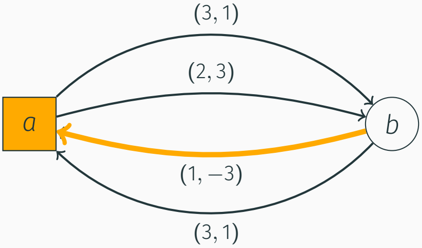


$((3,1)(3,1)(3,1)(3,1))^\omega \models \mathsf{P} \vee \underline{\mathsf{MP}}$

$(2,3)$

Example: $P \vee \underline{MP}$

P in 1^{st} coordinate $\quad$ $\underline{MP}$ in 2^{nd} coordinate

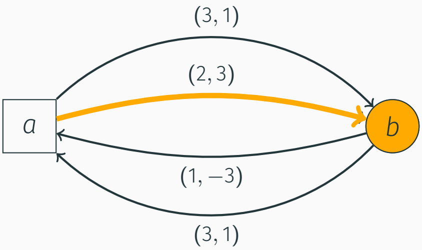


$((3,1)(3,1)(3,1)(3,1))^{\omega} \models P \vee \underline{MP}$

$(2,3)(1,-3)$

Example: $\mathsf{P} \vee \underline{\mathsf{MP}}$

P in 1^{st} coordinate $\quad$ $\underline{\mathsf{MP}}$ in 2^{nd} coordinate

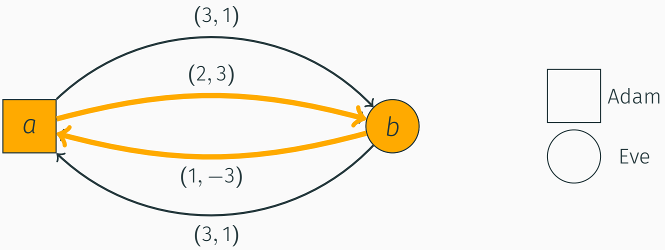


$((3,1)(3,1)(3,1)(3,1))^{\omega} \models \mathsf{P} \vee \underline{\mathsf{MP}}$

$(2,3)(1,-3)(2,3)$

Example: $\mathsf{P} \vee \underline{\mathsf{MP}}$

P in 1st coordinate $\qquad$ $\underline{\mathsf{MP}}$ in 2nd coordinate

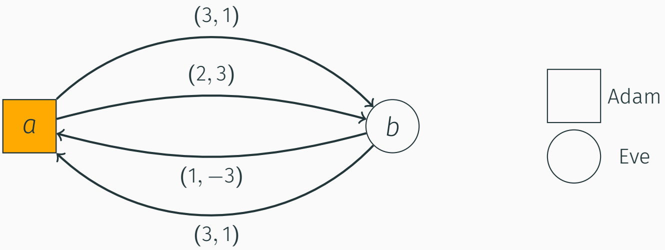


$((3,1)(3,1)(3,1)(3,1))^\omega \models \mathsf{P} \vee \underline{\mathsf{MP}}$

$((2,3)(1,-3)(2,3)(1,-3))^\omega \models \mathsf{P} \vee \underline{\mathsf{MP}}$

Example: $\mathsf{P} \vee \underline{\mathsf{MP}}$

P in 1^{st} coordinate $\underline{\mathsf{MP}}$ in 2^{nd} coordinate

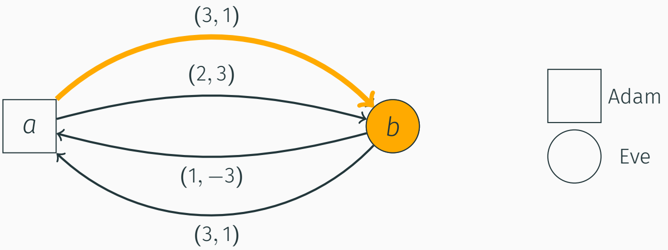


$((3,1)(3,1)(3,1)(3,1))^\omega \models \mathsf{P} \vee \underline{\mathsf{MP}}$

$((2,3)(1,-3)(2,3)(1,-3))^\omega \models \mathsf{P} \vee \underline{\mathsf{MP}}$

Example: $\mathsf{P} \vee \underline{\mathsf{MP}}$

P in 1st coordinate $\underline{\mathsf{MP}}$ in 2nd coordinate



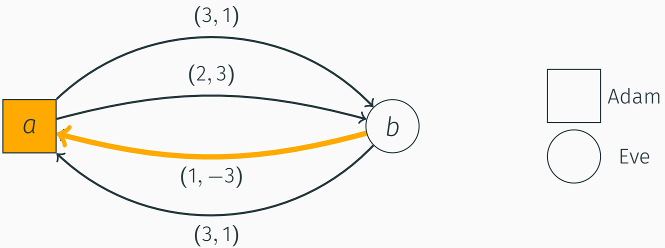
$((3,1)(3,1)(3,1)(3,1))^\omega \models \mathsf{P} \vee \underline{\mathsf{MP}}$

$((2,3)(1,-3)(2,3)(1,-3))^\omega \models \mathsf{P} \vee \underline{\mathsf{MP}}$

$(3,1)$

Example: $\text{P} \vee \underline{\text{MP}}$

P in 1st coordinate $\qquad$ $\underline{\text{MP}}$ in 2nd coordinate



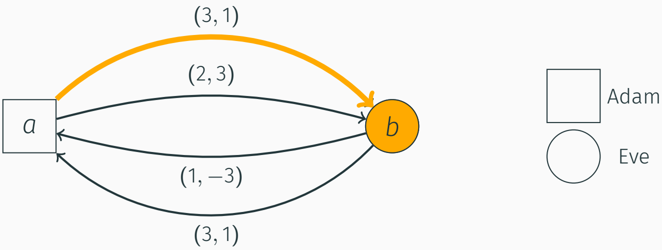
$((3,1)(3,1)(3,1)(3,1))^\omega \models \text{P} \vee \underline{\text{MP}}$

$((2,3)(1,-3)(2,3)(1,-3))^\omega \models \text{P} \vee \underline{\text{MP}}$

$(3,1)(1,-3)$

Example: $\mathsf{P} \vee \underline{\mathsf{MP}}$

P in 1^{st} coordinate $\qquad$ $\underline{\mathsf{MP}}$ in 2^{nd} coordinate



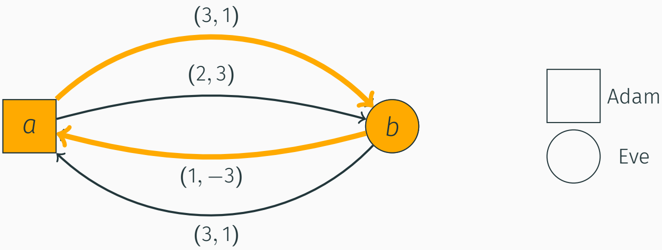
$((3,1)(3,1)(3,1)(3,1))^{\omega} \models \mathsf{P} \vee \underline{\mathsf{MP}}$

$((2,3)(1,-3)(2,3)(1,-3))^{\omega} \models \mathsf{P} \vee \underline{\mathsf{MP}}$

$(3,1)(1,-3)(3,1)$

Example: $\mathsf{P} \vee \underline{\mathsf{MP}}$

P in 1^{st} coordinate $\quad$ $\underline{\mathsf{MP}}$ in 2^{nd} coordinate



$((3,1)(3,1)(3,1)(3,1))^{\omega} \models \mathsf{P} \vee \underline{\mathsf{MP}}$

$((2,3)(1,-3)(2,3)(1,-3))^{\omega} \models \mathsf{P} \vee \underline{\mathsf{MP}}$

$((3,1)(1,-3)(3,1)(1,-3))^{\omega} \not\models \mathsf{P} \vee \underline{\mathsf{MP}}$

Why do we care?

- Synthesis of systems satisfying multiple constraints, qualitative or quantative

Why do we care?

- Synthesis of systems satisfying multiple constraints, qualitative or quantative
- `P` may represent *qualitative* constraints like reachability of a good behaviour, and `MP` may represent *quantative* constraints like power consumption.

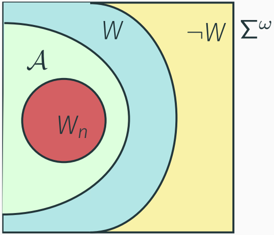
Separating automata for a winning condition W*

- Automaton $\mathcal{A}$ with safety acceptance condition such that

* Notion of Separating automata was introduced by Bojańczyk and Czerwiński, and this defintion was given by Colcombet and Fijalkow.

Separating automata for a winning condition W*

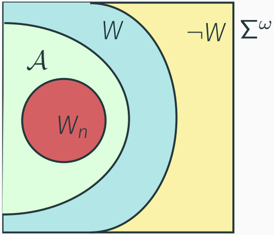
- Automaton $\mathcal{A}$ with safety acceptance condition such that
  - For all n-sized graphs satisfying W, $\mathcal{A}$ accepts all paths in the graph
  - $\mathcal{A}$ rejects all paths not satisfying W



* Notion of Separating automata was introduced by Bojańczyk and Czerwiński, and this defintion was given by Colcombet and Fijalkow.

Separating automata for a winning condition W*

- Automaton $\mathcal{A}$ with safety acceptance condition such that
  - For all n-sized graphs satisfying W, $\mathcal{A}$ accepts all paths in the graph
  - $\mathcal{A}$ rejects all paths not satisfying W



Theorem (Colcombet, Fijalkow 2019)

Let G be a game of size n with positional objective W and $\mathcal{A}$ be a (n, W)-separating automaton.
Then Eve has a strategy ensuring W if and only if she has a strategy winning the safety game $G \times \mathcal{A}$.

* Notion of Separating automata was introduced by Bojańczyk and Czerwiński, and this defintion was given by Colcombet and Fijalkow.

Separating automaton for $\vee_i \underline{\mathbf{MP}}_i$

Theorem (Chatterjee, Velner 2013)

There exists an algorithm for solving these games with complexity $\mathcal{O}(n^2 \cdot m \cdot k \cdot W \cdot (k \cdot n \cdot W)^{k^2+2k+1})$.

Theorem

There exists a separating automaton for $\vee_i \underline{\mathbf{MP}}_i$ of size $\mathcal{O}(n^k \cdot W^k)$, inducing an algorithm for solving these games with complexity $\mathcal{O}(m \cdot n^k \cdot W^k)$, where k is the number of $\underline{\mathbf{MP}}$ objectives.

Separating automaton for $\vee_i \underline{\textbf{MP}}_i$

Theorem (Chatterjee, Velner 2013)

There exists an algorithm for solving these games with complexity $\mathcal{O}(n^2 \cdot m \cdot k \cdot W \cdot (k \cdot n \cdot W)^{k^2+2k+1})$.

Theorem

There exists a separating automaton for $\vee_i \underline{\textbf{MP}}_i$ of size $\mathcal{O}(n^k \cdot W^k)$, inducing an algorithm for solving these games with complexity $\mathcal{O}(m \cdot n^k \cdot W^k)$, where k is the number of $\underline{\textbf{MP}}$ objectives.

Idea: Reduce the problem to construction of separating automata for strongly connected graphs, and then construct the later using the property that a strongly connected graph satisfying $\vee_i \underline{\textbf{MP}}_i$, satisfies $\underline{\textbf{MP}}$ in one of its coordinates.

Separating automaton for $P \vee \underline{MP}$

Theorem (Daviaud et al. 2018)

There exists a pseudo-quasi-polynomial time algorithm for solving games with objective $P \vee \underline{MP}$.

Separating automaton for P $\vee$ $\underline{\texttt{MP}}$

Theorem (Daviaud et al. 2018)

There exists a pseudo-quasi-polynomial time algorithm for solving games with objective P $\vee$ $\underline{\texttt{MP}}$.

Theorem

There exists a separating automaton for P $\vee$ $\underline{\texttt{MP}}$ of size $\mathcal{O}(d \cdot |\mathcal{A}_{\mathsf{P}}| \cdot |\mathcal{A}_{\underline{\texttt{MP}}}|)$, where d is the highest label in the parity coordinate.

Separating automaton for $\mathsf{P} \vee \underline{\mathsf{MP}}$

Theorem (Daviaud et al. 2018)

There exists a pseudo-quasi-polynomial time algorithm for solving games with objective $\mathsf{P} \vee \underline{\mathsf{MP}}$.

Theorem

There exists a separating automaton for $\mathsf{P} \vee \underline{\mathsf{MP}}$ of size $\mathcal{O}(d \cdot |\mathcal{A}_{\mathsf{P}}| \cdot |\mathcal{A}_{\underline{\mathsf{MP}}}|)$, where d is the highest label in the parity coordinate.

Idea: Keep simulating the separating automaton for $\underline{\mathsf{MP}}$, simulate P separating automaton with the maximum priority when the earlier rejects, and reject the run when the later rejects.

Separating automaton for $\mathsf{P} \vee \underline{\mathsf{MP}}$

Theorem (Daviaud et al. 2018)

There exists a pseudo-quasi-polynomial time algorithm for solving games with objective $\mathsf{P} \vee \underline{\mathsf{MP}}$.

Theorem

There exists a separating automaton for $\mathsf{P} \vee \underline{\mathsf{MP}}$ of size $\mathcal{O}(d \cdot |\mathcal{A}_{\mathsf{P}}| \cdot |\mathcal{A}_{\underline{\mathsf{MP}}}|)$, where d is the highest label in the parity coordinate.

Idea: Keep simulating the separating automaton for $\underline{\mathsf{MP}}$, simulate P separating automaton with the maximum priority when the earlier rejects, and reject the run when the later rejects.

Note: The separating automaton for $\mathsf{P} \vee \overline{\mathsf{MP}}$ is exactly the same.

Summary

- We give $\mathcal{O}(m \cdot n^k \cdot W^k)$ complexity algorithm for $\vee_i \underline{\texttt{MP}}_i$, with the separation approach, which is better than that given by Chatterjee and Velner (2013).

Summary

- We give $\mathcal{O}(m \cdot n^k \cdot W^k)$ complexity algorithm for $\vee_i \underline{\mathtt{MP}}_i$, with the separation approach, which is better than that given by Chatterjee and Velner (2013).
- We match the best known complexity of solving games with $\mathtt{P} \vee \underline{\mathtt{MP}}$ and $\mathtt{P} \vee \overline{\mathtt{MP}}$, i.e. pseudo-quasi-polynomial complexity, using separating automata.

Summary

- We give $\mathcal{O}(m \cdot n^k \cdot W^k)$ complexity algorithm for $\vee_i \underline{\mathtt{MP}}_i$, with the separation approach, which is better than that given by Chatterjee and Velner (2013).
- We match the best known complexity of solving games with $\mathtt{P} \vee \underline{\mathtt{MP}}$ and $\mathtt{P} \vee \overline{\mathtt{MP}}$, i.e. pseudo-quasi-polynomial complexity, using separating automata.
- Chatterjee and Velner (2013) solve the games with winning condition $\overline{\mathtt{MP}} \vee \overline{\mathtt{MP}}$, but it is still open to match the complexity with the separation approach.