Efficiently Testing Simon’s Congruence

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Subsequences

\[ w_{b\ a\ c\ b\ a\ a\ b\ a\ d\ a} \]
Subsequences
Subsequences

\[ w = b a c b a a b a d a \]
Subsequences
Subsequences

$w[i_1, i_2, i_3, \ldots, i_k]$

Subsequence

We call $w'$ a subsequence of length $k$ of a word $w$, where $|w| = n$, if there exist positions $1 \leq i_1 < i_2 < \ldots < i_k \leq n$, such that $w' = w[i_1]w[i_2]\cdots w[i_k]$.

Set of Subsequences of length $k$

Let $S\mathcal{F}_{\leq k}(i, w)$ denote the set of subsequences of length at most $k$ of $w[i : n]$. Accordingly, the set of subsequences of length at most $k$ of the entire word $w$ will be denoted by $S\mathcal{F}_{\leq k}(1, w)$.

Example: $S\mathcal{F}_2(1, abaca) = \{aa, ab, ac, ba, bc, ca\}$

$S\mathcal{F}_{\leq 2}(1, abaca) = \{a, b, c, aa, ab, ac, ba, bc, ca\}$
Simon’s Congruence

(i) Let $w, w' \in \Sigma^*$. We say that $w$ and $w'$ are equivalent under Simon’s congruence $\sim_k$ if $SF_{\leq k}(1, w) = SF_{\leq k}(1, w')$. 

(ii) Let $i, j \in w$. We define $i \sim_k j \ (w.r.t. w)$ if $w[i:n] \sim_k w[j:n]$, and we say that the positions $i$ and $j$ are $k$-equivalent.

(iii) A word $u$ of length $k$ distinguishes $w$ and $w'$ w.r.t. $\sim_k$ if $u$ occurs in exactly one of the sets $SF_{\leq k}(1, w)$ and $SF_{\leq k}(1, w')$. 
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Example: $w = abacab$, $w' = baacabba$
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$$SF_2(1, w) = \{aa, ab, ac, ba, bb, bc, ca, cb\}$$
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$SF_2(1, w') = \{aa, ab, ac, ba, bb, bc, ca, cb\}$

$SF_2(1, w) = SF_2(1, w') \Rightarrow w \sim_2 w'$
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Example: $w = abacab$, $w' = baacabba$

$$bbb \notin \mathcal{SF}_3(1, w), bbb \in \mathcal{SF}_3(1, w')$$
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(i) Let \( w, w' \in \Sigma^* \). We say that \( w \) and \( w' \) are equivalent under Simon’s congruence \( \sim_k \) if \( SF_{\leq k}(1, w) = SF_{\leq k}(1, w') \).

Example: \( w = abacab, w' = baacabba \)

\[ bbb \notin SF_3(1, w), bbb \in SF_3(1, w') \]

\[ SF_3(1, w) \neq SF_3(1, w') \Rightarrow w \sim_3 w' \]
Simon’s Congruence

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(iii) A word \( u \) of length \( k \) distinguishes \( w \) and \( w' \) w.r.t. \( \sim_k \) if \( u \) occurs in exactly one of the sets \( SF_{\leq k}(1, w) \) and \( SF_{\leq k}(1, w') \).

Example: \( w = abacab, w' = baacabba \)
Problem Definition

**SimK**
Given two words $s$ and $t$ over an alphabet $\Sigma$, with $|s| = n$ and $|t| = n'$, with $n \geq n'$, and a natural number $k$, decide whether $s \sim_k t$.

**MaxSimK**
Given two words $s$ and $t$ over an alphabet $\Sigma$, with $|s| = n$ and $|t| = n'$, with $n \geq n'$, find the maximum $k$ for which $s \sim_k t$. 
History

- Line of research originating in the PhD thesis of Imre Simon from 1972
- Long history of algorithm designs and improvements for associated problems. State of the art: \( \text{SimK} \) optimal linear time [DLT 2020] \( \text{MaxSimK} \) \( O(n \log n) \) time [DLT 2020].
- Today: an optimal linear-time algorithm for the \( \text{MaxSimK} \) problem.
Simon-tree
Equivalence Classes

\[
SF_k(i, w) \supset SF_k(l, w) \supset SF_k(j, w)
\]

▶ Splitting a word suffixwise into blocks of equivalence classes w.r.t. \( \sim_k \)

▶ If \( i \sim_k j \), then \( SF_k(i, w) = SF_k(l, w) = SF_k(j, w) \)
and we say that \( i, l, \) and \( j \) are in the same \( k \)-block

▶ \( \sim_{k+1} \) is a refinement of \( \sim_k \)

▶ Index \( i \) is a \( (k + 1) \)-splitting position if \( i \sim_k i + 1 \) but not \( i \sim_{k+1} i + 1 \)
Equivalence Classes

Use these properties to build a block structure for a word

1. \( i \sim_1 j \) iff \( \text{alph}(w[i : n]) = \text{alph}(w[j : n]) \) for any \( i, j \in w \)

→ We can go from right to left through the word and determine 1-splitting positions
Equivalence Classes

Use these properties to build a block structure for a word

1. \( i \sim_1 j \) iff \( \text{alph}(w[i : n]) = \text{alph}(w[j : n]) \) for any \( i, j \in w \)

\( \rightarrow \) We can go from right to left through the word and determine 1-splitting positions

\[
\begin{array}{cccccccc}
  & b & a & c & b & a & a & b & a & d & a \\
\end{array}
\]
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Equivalence Classes

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1. $i \sim_1 j$ iff $\text{alph}(w[i : n]) = \text{alph}(w[j : n])$ for any $i, j \in w$

   $\rightarrow$ We can go from right to left through the word and determine $1$-splitting positions

2. Split a $k$-block into $(k + 1)$-blocks by going from right to left through the block (without its last letter) and determine $(k + 1)$-splitting positions exactly as for $1$-splitting positions.

\[
\begin{array}{cccccccc}
 b & a & c & b & a & a & b & a & d & a \\
\end{array}
\]

1-blocks
Equivalence Classes

Use these properties to build a block structure for a word

1. \( i \sim_1 j \) iff \( \text{alph}(w[i:n]) = \text{alph}(w[j:n]) \) for any \( i, j \in w \)

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2. Split a \( k \)-block into \((k + 1)\)-blocks by going from right to left through the block (without its last letter) and determine \((k + 1)\)-splitting positions exactly as for 1-splitting positions.

1-blocks

\[
\begin{array}{cccccccc}
\text{w} & b & a & c & b & a & a & b & a & d & a \\
\end{array}
\]
Use these properties to build a block structure for a word

1. \( i \sim_1 j \) iff \( \text{alph}(w[i:n]) = \text{alph}(w[j:n]) \) for any \( i, j \in w \)
   
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2. Split a \( k \)-block into \( (k + 1) \)-blocks by going from right to left through the block (without its last letter) and determine \( (k + 1) \)-splitting positions exactly as for 1-splitting positions.

2-blocks

\[
\begin{array}{cccccccccccc}
  & b & a & c & b & a & a & b & a & d & a \\
\end{array}
\]
Equivalence Classes

Use these properties to build a block structure for a word

1. \( i \sim_1 j \) iff \( \text{alph}(w[i:n]) = \text{alph}(w[j:n]) \) for any \( i, j \in w \)
   → We can go from right to left through the word and determine 1-splitting positions

2. Split a \( k \)-block into \( (k + 1) \)-blocks by going from right to left through the block (without its last letter) and determine \( (k + 1) \)-splitting positions \textbf{exactly} as for 1-splitting positions.

2-blocks
Equivalence Classes

Use these properties to build a block structure for a word

1. $i \sim_1 j$ iff $\text{alph}(w[i : n]) = \text{alph}(w[j : n])$ for any $i, j \in w$

   $\rightarrow$ We can go from right to left through the word and determine $1$-splitting positions

2. Split a $k$-block into $(k + 1)$-blocks by going from right to left through the block (without its last letter) and determine $(k + 1)$-splitting positions exactly as for $1$-splitting positions.

\[
\begin{array}{cccccccc}
  w & b & a & c & b & a & a & b & a & d & a \\
\end{array}
\]
Equivalence Classes

Use these properties to build a block structure for a word

1. $i \sim_1 j$ iff $\text{alph}(w[i : n]) = \text{alph}(w[j : n])$ for any $i, j \in w$
   → We can go from right to left through the word and determine 1-splitting positions

2. Split a $k$-block into $(k + 1)$-blocks by going from right to left through the block (without its last letter) and determine $(k + 1)$-splitting positions **exactly** as for 1-splitting positions.

3-blocks

\[
\begin{array}{cccccccc}
  w & b & a & c & b & a & a & b & a & d & a \\
  \hline
  & & & & & & & & & &
\end{array}
\]
Simon-tree Definition

- New data structure: Simon-tree
- Represents presented block structure
- Efficiently partition positions of a given word
- Construction takes linear time
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bacbaabada

$[1 : 10]$

bac

$[1 : 3]$

b

$[1]$

a

$[2]$

c

$[3]$

baab

$[4 : 7]$

b

$[4]$

aa

$[5 : 6]$

b

$[7]$

a

$[8]$

d

$[9]$

a

$[10]$

$[k = 0]$

$[k = 1]$

$[k = 2]$

$[k = 3]$
Simon-tree Definition

The *Simon-tree* $T_w$ associated to the word $w$, with $|w| = n$, is an ordered rooted tree. The nodes represent $k$–blocks of $w$, for $0 \leq k \leq n$, and are defined recursively.

▶ The root corresponds to the 0-block of the word $w$, i.e., the interval $[1 : n]$.

▶ For $k > 1$ and for a node $b$ on level $k - 1$, which represents a $(k - 1)$-block $[i : j]$ with $i < j$, the children of $b$ are exactly the blocks of the partition of $[i : j]$ in $k$-blocks, ordered decreasingly by their starting position.

▶ For $k > 1$, each node on the level $k - 1$ which represents a $(k - 1)$-block $[i : i]$ is a leaf.
Algorithm: Build the Simon-tree right to left as the word is traversed right to left. Only the leftmost branch is edited during construction.

1. The level (block), where a new position/letter should be assigned to (resp., belongs to), is computed efficiently.
2. Insert the new position/letter into the tree by moving up the leftmost branch from leaf to root.
3. Close traversed blocks on the path until the level for the new position/letter is reached.
4. Insert the new position/letter as a leftmost child on its corresponding level.
### Simon-tree Construction

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$k = 0$
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$k = 1$
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$ k = 0 $

$ k = 1 $
## Simon-tree Construction

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\[ \begin{array}{c}
\ldots \$ \\
11] \\
\ldots \text{d} \\
9] \\
[10] \\
k = 0 \\
\ldots \text{a} \\
[10] \\
k = 1 \\
\$ \\
\end{array} \]
Simon-tree Construction

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...$

11

...d

9

[9]

a

[10]

[11]

$  

k = 0

k = 1
Simon-tree Construction

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$X_{4,5,\infty,7,6,8,\infty,10,\infty,\infty,\infty}$

$k = 0$

$k = 1$
Simon-tree Construction

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$\ldots$ $\ldots d$

$9 [10] [11]$

$k = 1$

$\ldots a$

$8 [9]$
Simon-tree Construction

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11

$\ldots$ d 

9

[10] 

[11] 

a 

$\$ 

k = 0

$\ldots$ a 

8

[9] 

d 

9

[10] 

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$k = 1$

$k = 2$
Simon-tree Construction

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$k = 1$

$k = 2$
Simon-tree Construction

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Simon-tree Construction

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\[
\begin{array}{cccccccccc}
11] & \ldots$ & \\
7] & \ldots b & \\
8:9] & \text{ad} & \\
[9] & \text{d} & \\
[8] & \text{a} & \\
[7] & \text{b} & \\
\end{array}
\]

\[
k = 0
\]

\[
k = 1
\]

\[
k = 2
\]
Simon-tree Construction

<table>
<thead>
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<th>position</th>
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$k = 0$

\[
\begin{array}{c}
\ldots\$
\end{array}
\]

\[
\begin{array}{c}
11]
\end{array}
\]

$k = 1$

\[
\begin{array}{c}
\ldots b
\end{array}
\]

\[
\begin{array}{c}
\ldots \text{ad}
\end{array}
\]

\[
\begin{array}{c}
\ldots a
\end{array}
\]

\[
\begin{array}{c}
\ldots $
\end{array}
\]

$k = 2$

\[
\begin{array}{c}
\ldots b
\end{array}
\]

\[
\begin{array}{c}
\ldots [8:9]
\end{array}
\]

\[
\begin{array}{c}
\ldots [10]
\end{array}
\]

\[
\begin{array}{c}
\ldots [11]
\end{array}
\]
Simon-tree Construction

\[
\begin{array}{cccccccccccc}
\text{position} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
\hline
w & b & a & c & b & a & a & b & a & d & a & \$
\hline
X & 4 & 5 & \infty & 7 & 6 & 8 & \infty & 10 & \infty & \infty & \infty
\end{array}
\]

\[
\begin{array}{cccccccccccc}
\text{tree structure}
\end{array}
\]

\[
\begin{array}{cccccccccccc}
\text{tree structure}
\end{array}
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\begin{array}{cccccccccccc}
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Simon-tree Construction

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- $k = 0$

```
  ...$
   11]
```

- $k = 1$

```
  ...b
  7]
```

```
  ...a
  [6]
```

```
  b
  [7]
```

```
  a
  [8]
  [9]
```

```
  d
  [10]
  [11]
```

- $k = 2$

```
  a
  [6]
```

```
  ad
  [8:9]
```

```
  a
  [10]
```

```
  $$
  [11]
```

```
  k = 0
```

```
  k = 1
```

```
  k = 2
```
Simon-tree Construction

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$k = 0$

$k = 1$

$k = 2$
Simon-tree Construction

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$k = 0$

$k = 1$

$k = 2$

$k = 3$
Simon-tree Construction

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$k = 0$

$k = 1$

$k = 2$

$k = 3$
Simon-tree Construction

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\[
\begin{align*}
\text{position} & \quad 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
\text{w} & \quad b & a & c & b & a & a & b & a & d & a & $ \\
\text{X} & \quad 4 & 5 & \infty & 7 & 6 & 8 & \infty & 10 & \infty & \infty & \infty \\
\end{align*}
\]

$\ldots$ $\infty$

$k = 0$

$\ldots b$

$\infty$

$k = 1$

$\ldots a$

$\infty$

$k = 2$

$\ldots a$

$\infty$

$k = 3$
Simon-tree Construction

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## Simon-tree Construction

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11]

...b

7]

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Simon-tree Construction

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$ k = 1 

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**Simon-tree Construction**

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bacbaabada$

[1 : 11]

bac

[1:3]

b
[1]

a
[2]

c
[3]

baab

[4 : 7]

b
[4]

a
[5]

aa
[5:6]

b
[6]

ad

[8:9]

a
[8]

$  
[11]  
k = 0

[k = 1

bacbaabada$  

[1 : 11]

bac

[1:3]

b
[1]

a
[2]

c
[3]

baab

[4 : 7]

b
[4]

a
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aa
[5:6]

b
[6]

ad

[8:9]

a
[8]

$  
[11]  
k = 1

bacbaabada$  

[1 : 11]

bac

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aa
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ad

[8:9]

a
[8]

$  
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k = 2

bacbaabada$  

[1 : 11]

bac

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a
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c
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baab

[4 : 7]

b
[4]

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aa
[5:6]

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[6]

ad

[8:9]

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[8]

$  
[11]  
k = 3
Simon-tree Construction

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$X_4 5 \infty 7 6 8 \infty 10 \infty \infty \infty$

$k = 0$

$[1:10]$

$k = 1$

$k = 2$

$k = 3$
Simon-tree Construction

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$X = \{4, 5, \infty, 7, 6, 8, \infty, 10, \infty, \infty, \infty\}$

- $k = 0$
- $k = 1$
- $k = 2$
- $k = 3$
\begin{verbatim}

\begin{tabular}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline
\textbf{position} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
\hline
w & b & a & c & b & a & a & b & a & d & a & $\$ \\
\hline
X & 4 & 5 & $\infty$ & 7 & 6 & 8 & $\infty$ & 10 & $\infty$ & $\infty$ & $\infty$ \\
\hline
\end{tabular}

\end{verbatim}

bacbaabada

\begin{verbatim}

[1 : 10]

bac
[1:3]

[1] b
[2] a
[3] c

baab
[4 : 7]

[4] b
[5:6] aa
[7] b

ad
[8:9]

[8] a
[9] d

[10]

[5] a
[6] a

k = 0

k = 1

k = 2

k = 3

\end{verbatim}
So far:
structure for one word representing the equivalence classes w.r.t. $\sim_k$

Now:
set two words in relation to each other by using their respective Simon-trees

**MaxSimK**

Given two words $s$ and $t$ over an alphabet $\Sigma$, with $|s| = n$ and $|t| = n'$, with $n \geq n'$, find the maximum $k$ for which $s \sim_k t$. 
Transform the words $s$ and $t$ into Simon-trees as shown.

Use the tree structure to connect equivalent nodes of the two words.
Connecting Two Simon-trees

- Transform the words $s$ and $t$ into Simon-trees as shown.
- Use the tree structure to connect equivalent nodes of the two words.

**S-Connection**
The $k$-node $a$ of $T_s$ and the $k$-node $b$ of $T_t$ are S-connected (i.e., the pair $(a, b)$ is in the S-connection) if and only if $s[i:n] \sim_k t[j:n']$ for all positions $i$ in block $a$ and positions $j$ in block $b$. 
Starting from a larger relation (P-Connection) which contains the S-Connection, and refine it.

- The 0-nodes of $T_s$ and $T_t$ are P-connected.
- For all levels $k$ of $T_s$, if the explicit or implicit $k$-nodes $a$ and $b$ (from $T_s$ and $T_t$, respectively) are P-connected, then the $i^{th}$ child of $a$ is P-connected to the $i^{th}$ child of $b$, for all $i$.
- No other nodes are P-connected.
From P-Connection to S-Connection
From P-Connection to S-Connection

How to refine the P-Connection:

- Let $k \geq 1$. Let $a, b$ be $k$-blocks in the word $t$, resp. $s$, with $a \sim_k b$.
- Let $a'$ be child of $a$, $b'$ be child of $b$.
- $a' \sim_{k+1} b'$ if and only if there exists a letter $x$ such that $s[\text{next}(a', x) + 1 : n] \sim_k t[\text{next}(b', x) + 1 : n']$. 
From P-Connection to S-Connection
From P-Connection to S-Connection

From P-Connection to S-Connection
From P-Connection to S-Connection
From P-Connection to S-Connection
From P-Connection to S-Connection
From P-Connection to S-Connection
From P-Connection to S-Connection

![Diagram showing the transition from P-Connection to S-Connection with examples of abacab and baacabba strings and their corresponding connections.](Diagram.png)
Solution of $\text{MaxSimK}$: last level $k$ where the $k$-blocks containing position 1 of the input words are equivalent.

- Distinguishing word can be obtained.
- By efficiently using union-find and split find data structures the algorithm achieves an optimal linear runtime.
Additional Notes and Analysis

- Solution of $\text{MaxSimK}$: last level $k$ where the $k$-blocks containing position 1 of the input words are equivalent.
- Distinguishing word can be obtained.
- By efficiently using union-find and split find data structures the algorithm achieves an optimal linear runtime.
Solution of $\text{MaxSimK}$: last level $k$ where the $k$-blocks containing position 1 of the input words are equivalent.

- Distinguishing word can be obtained.
- By efficiently using union-find and split find data structures the algorithm achieves an optimal linear runtime.

Thank you!