Positionality and strategy improvement for continuous payoffs

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Infinite games on finite graphs

Fix a function $\phi: C^\omega \to \mathbb{R}$ (called a *payoff*). A $\phi$-game consists of:

- $c, d, e, f \in C$
- Min and Max are shifting a pebble (Min in $\triangle$-nodes and Max in $\square$-nodes) along the edges. Infinitely many shifts.
- Zero sum: Min pays Max a fine of size $\phi(c_1c_2c_3\ldots)$, where $c_1, c_2, c_3, \ldots$ are colors along trajectory of the pebble.

Definition

A payoff $\phi$ is *positional* if in all $\phi$-games players can play optimally via a positional strategy.
Continuous payoffs

**Definition**
A payoff \( \phi : C^\omega \to \mathbb{R} \) is **continuous** if for any \( \alpha \in C^\omega \) and for any infinite sequence \( \beta_1, \beta_2, \beta_3, \ldots \in C^\omega \) the following holds. Assume that for any \( i \in \mathbb{N} \) we have that \( \alpha \) and \( \beta_i \) coincide in the first \( i \) elements. Then

\[
\phi(\alpha) = \lim_{i \to \infty} \phi(\beta_i)
\]

Can be defined by the cylinder topology, which is compact.

**Examples:** (multi)discounted payoff is continuous, Parity and Mean Payoff are not.
Characterizing positional payoffs

A payoff $\phi : C^\omega \rightarrow \mathbb{R}$ is called **prefix-monotone** if there are no $x, y \in C^*$ and $\alpha, \beta \in C^\omega$ such that

$$\phi(x\alpha) > \phi(x\beta), \quad \phi(y\alpha) < \phi(y\beta).$$

**Theorem**

Let $\phi : C^\omega \rightarrow \mathbb{R}$ be a continuous payoff. Then $\phi$ is positional if and only if $\phi$ is prefix-monotone.

![Diagram](image)

**Figure:** The “only if” part.
What else can be said

Generalizing some results for (multi)discounted payoffs.

- strategy improvement (all continuous positional payoffs)
- LP-type problems and subexponential randomized algorithms (all continuous positional payoffs).
- Strong bounds on strategy improvement (for generalized or non-linear discounted payoff).

What about stochastic games?

- Continuous + stochastically positional $\implies$ (multi)discounted.
Thank you!